TRIGONOMETRY

REQUIRED FOR THE ADDITIONAL SUBJECTS FOR HONOURS

AT THE

PREVIOUS EXAMINATION.

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ACCORDING TO THE NEW SCHEME SANCTIONED BY THE SENATE, JUNE 1865.

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TRIGONOMETRY.

CORRECTIONS AND ALTERATIONS.

Page 1, Art. 2, line 2, after "intersecting" insert ***at right angles***.
Page 5, line 15, for "measure of D" read ***measure of A***.
Pages 5 and 6, for "D" substitute l.
Page 13, line 2, for "PM''" read $P'M'$.
Page 27, line 11, for "prove them" read ***prove those in Art.*** 27.
Page 30, line 19, omit "smaller."

PLANE TRIGONOMETRY.

CHAPTER I.

CONVENTION RESPECTING THE SIGNS + AND −. CIRCUMFERENCE AND AREA OF A CIRCLE. MODES OF MEASURING ANGLES.

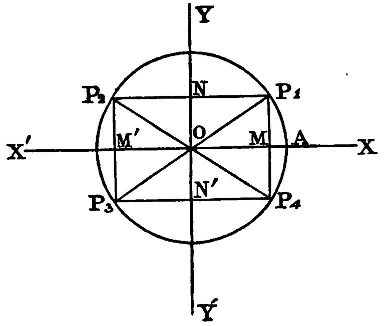
1. The following Elementary Treatise on Plane Trigonometry contains "the modes of measuring angles, trigonometrical ratios, functions of two angles, and the properties of triangles," besides a Chapter on the solution of Triangles.

2. Let $X'X$, $Y'Y$ be two indefinite straight lines intersecting in O and dividing the plane of the paper into the four quadrants XOY, YOX', $X'OY'$ and $Y'OX$, which are called respectively the *first*, *second*, *third*, and *fourth* quadrants.

If OX and OY be considered the positive directions, then OX' and OY' will be the negative directions.

Thus, if P_1M and P_3M' be perpendicular to $X'X$ and P_1N, and P_3N' to $Y'Y$, then OM or P_1N is positive, and also ON or P_1M; whilst OM' or P_3N' and ON' or P_3M' are negative.

In other words, perpendicular distances from $X'X$ and *above* it are positive, *below* it, negative. Similarly perpendicular distances from YY' and to the *right* of it are positive, to the *left* of it, negative.



3. Suppose (Fig. of Art. 2) OA at first to coincide with OX, and to revolve from right to left. It will thus successively occupy the positions OP_1, OP_2, OP_3, OP_4, after having described the following angular spaces, viz.:

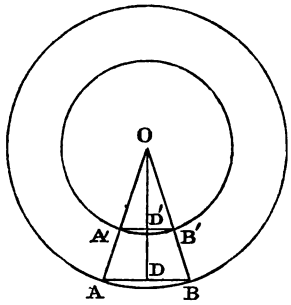
AOP_1 lying between zero and one right angle.
AOP_2 .... one and two right angles.
AOP_3 .... two and three ....
AOP_4 .... three and four ....

If OA continue its revolution, it will again coincide with OX after it has swept through four right angles, and so on.

Again, if OA revolve in the opposite direction from OX towards OY', as through the *acute* angle AOP_4, the angles thus swept through will be negative.

4. *The circumference of a circle bears a constant ratio to its diameter, and its area bears the same constant ratio to the square on its radius.* (See Euclid XII. 2.)

Let AB, $A'B'$ be two sides of regular polygons of n sides inscribed in any two circles which have the same centre O. Let C, r, A be the circumference, radius and area of the outer circle; and C', r' the circumference and radius of the inner circle.



Draw $OD'D$ perpendicular to $A'B'$ and AB.

Then from the similar triangles AOB, $A'OB'$

we have $$\frac{AB}{AO} = \frac{A'B'}{A'O},$$

that is $$\frac{AB}{r} = \frac{A'B'}{r'},$$

therefore $$\frac{n \,.\, AB}{r} = \frac{n \,.\, A'B'}{r'}.$$

Now, $n \,.\, AB$ and $n \,.\, A'B'$ are the perimeters of the two polygons, and when n is indefinitely increased and therefore the length of each side indefinitely diminished, these perimeters ultimately coincide with the circumferences of the circles; in this case

therefore $$\frac{n \,.\, AB}{r} = \frac{n \,.\, A'B'}{r'}$$

becomes $$\frac{C}{r} = \frac{C'}{r'}.$$

The constant ratio $\frac{C'}{r'}$ is usually denoted by 2π; hence we have $$\frac{C}{r} = 2\pi,$$

$$\therefore C = 2\pi r \ldots (1).$$

Again, $\Delta\, AOB = \frac{1}{2} OD \,.\, AB$,

$\therefore$ polygon inscribed in outer circle $= n \,.\, \Delta\, AOB$,

$$= \tfrac{1}{2} OD \,.\, n \,.\, AB;$$

but when n is indefinitely increased, OD ultimately $= r$ and $n \,.\, AB$ ultimately $= C$, or $2\pi r$ by (1);

also the polygon ultimately coincides with the circle,

$$\therefore\ A = \tfrac{1}{2} r \,.\, 2\pi r = \pi r^2 \ldots\ldots(2).$$

N.B. $\pi = 3{\cdot}14159265$ true to 8 places of decimals, but in most cases it is sufficient to take $\pi = 3{\cdot}1416$.

$\pi = \frac{22}{7}$ is too great by $\frac{1}{791}$ very nearly.

5. *Sexagesimal or English method of measuring angles.*

If a right angle be divided into 90 equal parts, each part is called a *degree*, the 60th part of a degree is called a *minute*, and the 60th part of a minute is called a *second*, smaller angles being expressed as decimals of a second.

$35^\circ\, 19'\, 12''$ is read 35 degrees, 19 minutes, 12 seconds.

Since (Euclid VI. 33) in the same circle angles at the centre are as the arcs which subtend them, therefore if the circumference of a circle be divided into 360 equal parts, each part will subtend an angle of one degree at the centre.

The arc which subtends any angle at the centre is also called an arc of as many degrees as the angle contains.

6. *Centesimal or French method of measuring angles.*

If a right angle be divided into 100 equal parts, each part is called a *grade*, the 100th part of a grade is called a *minute*, and the 100th part of a minute is called a *second.*

$35^g\, 19^{\backprime}\, 12^{\backprime\backprime}$ is read 35 grades, 19 minutes, 12 seconds.

7. *Circular or Theoretical measure of angles.*

The ratio $\frac{\text{arc}}{\text{radius}}$ is called the *circular measure* of the angle which the arc of the circle subtends at the centre.

The circular measure of any angle is independent of the length of the radius of the circle.

For $\frac{2\pi r}{4}$, that is, $\frac{1}{2}\pi r$, is the fourth part of the circumference, and subtends a right angle at the centre;

$\therefore \frac{1}{2}\pi r \div r = \frac{1}{2}\pi$ is the circular measure of a right angle; and generally if A be any angle at the centre expressed in degrees, and D the subtending arc, since the whole circumference subtends an angle of 360°, we have (Euclid VI. 33)

$$A : 360 :: D : 2\pi r,$$

$$\therefore D = \frac{A}{360} \cdot 2\pi r.$$

But the circular measure of D

$$= \frac{D}{r},$$

$$= \frac{A}{360} \cdot \frac{2\pi r}{r},$$

$$= \frac{A}{180} \cdot \pi,$$

$$= A \div \frac{180}{\pi} \ldots (1),$$

which is independent of r. Q.E.D.

COR. Since to express an angle in terms of a given angular unit only means to find how often that unit is contained in the given angle, we see from (1) *that* $\frac{180^\circ}{\pi}$ *is the unit of angular magnitude in the circular measure of an angle.*

Let $D=r$, then $\frac{D}{r}$ becomes $\frac{r}{r}$ or 1, and we have from (1)

$$1=A\cdot\frac{\pi}{180^\circ}\therefore A=\frac{180^\circ}{\pi},$$

$=57^\circ{\cdot}29577951,$

$=206264''{\cdot}8,$

or $206265''$ nearly,

= *the angle subtended at the centre by an arc equal to radius,*

or = *angular unit in the circular measure.*

8. Angles expressed in degrees are generally denoted by the Roman letters A, B, C, &c., and expressed in circular measure by the Greek letters α, β, γ, &c.

For the same angle we have therefore the relations

$$\alpha=A^\circ\div\frac{180^\circ}{\pi}=A^\circ\cdot\frac{\pi}{180^\circ},$$

$$\therefore A^\circ=\alpha\cdot\frac{180^\circ}{\pi}=\alpha\times 57^\circ{\cdot}29577951.$$

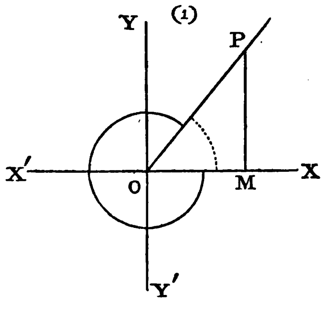
CHAPTER II.

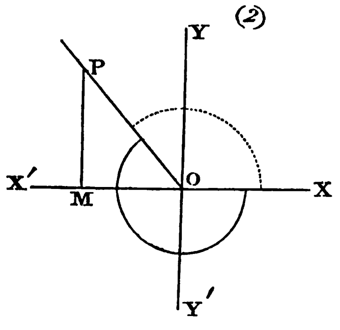
DEFINITIONS AND RELATIONS OF TRIGONOMETRICAL RATIOS. CHANGES IN THE SIGN AND MAGNITUDE OF THE TRIGONOMETRICAL RATIOS.

9. *Trigonometrical abbreviations.*

sin for sine,	sec for secant,
cos for cosine,	cosec for cosecant,
tan for tangent,	vers for versed sine,
cot for cotangent,	covers for coversed sine.

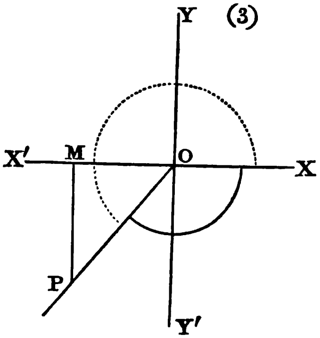
10. *Trigonometrical ratios or functions.*

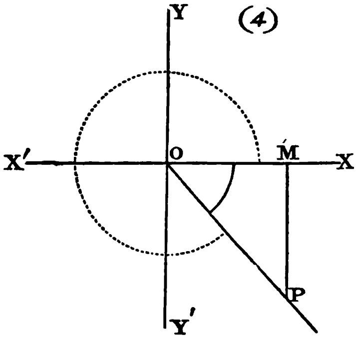




Let the two indefinite straight lines $X'X$, $Y'Y$ meet at right angles in O, and call $X'X$ *the initial line.*

Suppose OP to revolve, either in the positive or negative direction, from its initial position in OX to any other position OP.





From any point P in OP draw PM perpendicular to $X'X$; then if we put the angle $POX = A$, we have the following definitions:

$$\sin A = \frac{PM}{OP}, \qquad \cos A = \frac{OM}{OP},$$

$$\tan A = \frac{PM}{OM}, \qquad \cot A = \frac{OM}{PM},$$

$$\sec A = \frac{OP}{OM}, \qquad \operatorname{cosec} A = \frac{OP}{PM},$$

$$\operatorname{vers} A = 1 - \cos A, \qquad \operatorname{covers} A = 1 - \sin A.$$

$\cos^2 A$ means $(\cos A)^2$ or the square of $\cos A$, and so in similar cases.

In the above figures A denotes any of the positive angles subtended at O by the dotted arcs or any of the negative angles subtended at O by the other arcs.

The learner will bear in mind that the positive angles are measured from OX towards the left (or OY), and the negative angles towards the right (or OY').

The revolving line OP is always considered positive.

In the first quadrant or in the angle YOX, PM and OM are both positive.

In the second quadrant or in the angle YOX', PM is positive and OM negative.

In the third quadrant or in the angle $Y'OX'$, PM and OM are both negative.

In the fourth quadrant or in the angle $Y'OX$, PM is negative and OM positive.

When OP revolves, either in the positive or negative direction, into the same position again, it is clear that *all* the trigonometrical ratios are the same as before.

Hence, *if an angle be increased or diminished by four right angles, or any multiple of four right angles, no change will take place in the trigonometrical ratios of the angle.*

11. From the expressions in the last Article the following important results are deduced:

$$\sin A \,.\, \operatorname{cosec} A = \frac{PM}{OP} \cdot \frac{OP}{PM} = 1 \text{ or } \operatorname{cosec} A = \frac{1}{\sin A},$$

$$\cos A \,.\, \sec A = \frac{OM}{OP} \cdot \frac{OP}{OM} = 1 \text{ or } \sec A = \frac{1}{\cos A},$$

$$\tan A \,.\, \cot A = \frac{PM}{OM} \cdot \frac{OM}{PM} = 1 \text{ or } \cot A = \frac{1}{\tan A},$$

$$\frac{\sin A}{\cos A} = \frac{PM}{OP} \div \frac{OM}{OP} = \frac{PM}{OM} = \tan A,$$

$$\therefore \frac{\cos A}{\sin A} = \frac{1}{\tan A} = \cot A.$$

From the right-angled triangle POM we have

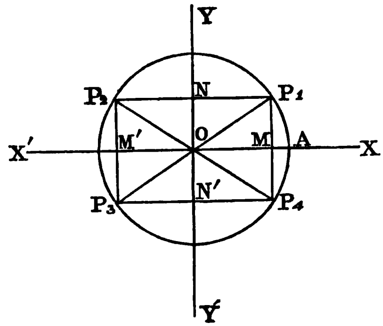
$$\cos^2 A + \sin^2 A = \frac{OM^2}{OP^2} + \frac{PM^2}{OP^2} = \frac{OP^2}{OP^2} = 1,$$

$$\sec^2 A = \frac{OP^2}{OM^2} = \frac{OM^2 + PM^2}{OM^2} = 1 + \left(\frac{PM}{OM}\right)^2 = 1 + \tan^2 A,$$

$$\operatorname{cosec}^2 A = \frac{OP^2}{PM^2} = \frac{PM^2 + OM^2}{PM^2} = 1 + \left(\frac{OM}{PM}\right)^2 = 1 + \cot^2 A.$$

12. *To prove that* $\sin(-A) = -\sin A$ *and* $\cos(-A) = \cos A$, *and to deduce the other trigonometrical ratios of* $-A$.

Let $P_1OM = A$, and $P_4OM = -A$.



So that the right-angled triangles P_1OM and P_4OM are *geometrically* equal in all respects; but P_4M being measured *below* $X'X$ is negative, and therefore $P_4M = -P_1M$.

By definition

$$\sin(-A) = \frac{P_4M}{OP_4} = \frac{-P_1M}{OP_1} = -\sin A,$$

$$\cos(-A) = \frac{OM}{OP_4} = \frac{OM}{OP_1} = \cos A.$$

The proof is exactly the same in whatever quadrant OP_1 may be situated. This remark is applicable to the next four Articles, but the learner should make figures for the cases in which OP is situated in the three other quadrants.

Hence also

$$\tan(-A) = \frac{\sin(-A)}{\cos(-A)} = \frac{-\sin A}{\cos A} = -\tan A,$$

$$\therefore \cot(-A) = -\cot A,$$

$$\sec(-A) = \frac{1}{\cos(-A)} = \frac{1}{\cos A} = \sec A,$$

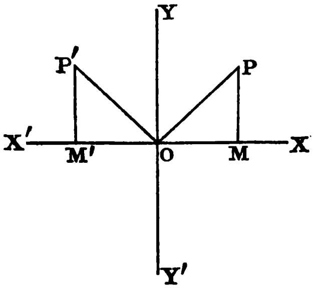
$$\text{cosec}(-A)=\frac{1}{\sin(-A)}=\frac{1}{-\sin A}=-\text{cosec}A,$$
$$\text{vers}(-A)=1-\cos(-A)=1-\cos A,$$
$$\text{covers}(-A)=1-\sin(-A)=1+\sin A.$$

13. *To prove that*

$$\sin(180°-A)=\sin A \text{ and } \cos(180°-A)=-\cos A.$$

Let $\angle POX=A$ and make the $\angle P'OX'=A$, then
$$P'OX=180°-A.$$

Make $OP'=OP$, then the right-angled triangles POM and $P'OM'$ are *geometrically* equal in all respects.



$$\therefore \sin(180°-A)=\sin P'OX=\frac{P'M'}{OP'}=\frac{PM}{OP}=\sin POM=\sin A,$$

$$\cos(180°-A)=\cos P'OX=\frac{OM'}{OP'}=\frac{-OM}{OP}=-\cos POM=-\cos A.$$

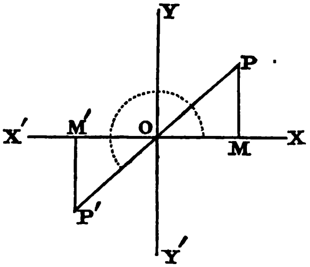
Q.E.D.

$180°-A$ is called the *supplement* of A.

Cor. Hence

$$\tan(180°-A)=\frac{\sin(180°-A)}{\cos(180°-A)}=\frac{\sin A}{-\cos A}=-\tan A,$$
$$\therefore \cot(180°-A)=-\cot A.$$
$$\sec(180°-A)=\frac{1}{\cos(180°-A)}=\frac{1}{-\cos A}=-\sec A,$$
$$\text{cosec}(180°-A)=\frac{1}{\sin(180°-A)}=\frac{1}{\sin A}=\text{cosec}A.$$

14. *To prove that*



$\sin(180° + A) = -\sin A$ *and*
$\cos(180° + A) = -\cos A$.

Let $POM = A$.

Produce OP to P' and make $OP' = OP$.

Then the right-angled triangles POM and $P'OM'$ are *geometrically* equal in all respects. Also the angle XOP' measured from OX in the positive direction (and subtended by the dotted arc) $= 180° + A$.

$$\therefore \sin(180° + A) = \frac{P'M'}{OP'} = \frac{-PM}{OP} = -\sin A,$$

$$\cos(180° + A) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A. \quad \text{Q.E.D.}$$

Cor. Hence

$$\tan(180° + A) = \frac{\sin(180° + A)}{\cos(180° + A)} = \frac{-\sin A}{-\cos A} = \tan A,$$

$$\therefore \cot(180° + A) = \cot A.$$

$$\sec(180° + A) = \frac{1}{\cos(180° + A)} = \frac{1}{-\cos A} = -\sec A,$$

$$\text{cosec}(180° + A) = \frac{1}{\sin(180° + A)} = \frac{1}{-\sin A} = -\text{cosec}A.$$

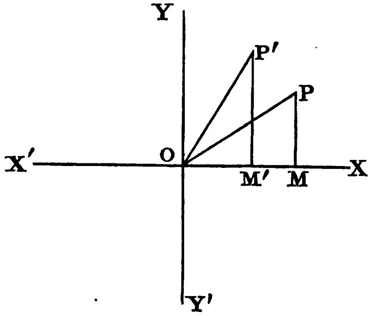
15. *To prove that*

$\sin(90° - A) = \cos A$ *and* $\cos(90° - A) = \sin A$.

Let $\angle POX = A$ and make $\angle YOP' = A$, so that $P'OM' = 90° - A$. Make $OP' = OP$.

The right-angled triangles POM and $P'OM'$ are clearly equal in all respects, and $P'M' = OM$, $OM' = PM$.

$$\therefore \sin(90^\circ - A) = \sin P'OM' = \frac{PM'}{OP'} = \frac{OM}{OP} = \cos A,$$



$$\cos(90^\circ - A) = \cos P'OM' = \frac{OM'}{OP'} = \frac{PM}{OP} = \sin A. \quad \text{Q.E.D.}$$

$(90^\circ - A)$ is called the *complement* of A.

COR. Hence

$$\tan(90^\circ - A) = \frac{\sin(90^\circ - A)}{\cos(90^\circ - A)} = \frac{\cos A}{\sin A} = \cot A,$$

$$\therefore \cot(90^\circ - A) = \tan A.$$

$$\sec(90^\circ - A) = \frac{1}{\cos(90^\circ - A)} = \frac{1}{\sin A} = \operatorname{cosec} A,$$

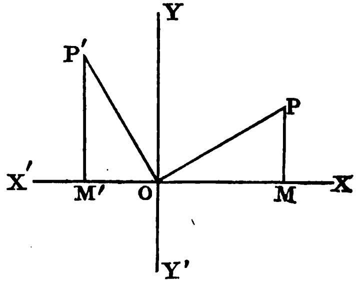
$$\operatorname{cosec}(90^\circ - A) = \frac{1}{\sin(90^\circ - A)} = \frac{1}{\cos A} = \sec A.$$

16. *To prove that*

$$\sin(90^\circ + \text{A}) = \cos\text{A} \text{ and } \cos(90^\circ + \text{A}) = -\sin\text{A}.$$

Let $\angle POX = A$, and make $YOP' = A$, so that

$$P'OX = 90^\circ + A.$$



Also make $OP' = OP$, then the right-angled triangles POM and $P'OM'$ are *geometrically* equal in all respects, and $OM' = -PM$, $P'M' = OM$.

$$\therefore \sin(90^\circ + A) = \sin P'OX = \frac{P'M'}{OP'} = \frac{OM}{OP} = \cos A.$$

$$\cos(90^\circ + A) = \cos P'OX = \frac{OM'}{OP'} = \frac{-PM}{OP} = -\sin A. \quad \text{Q.E.D.}$$

COR. Hence

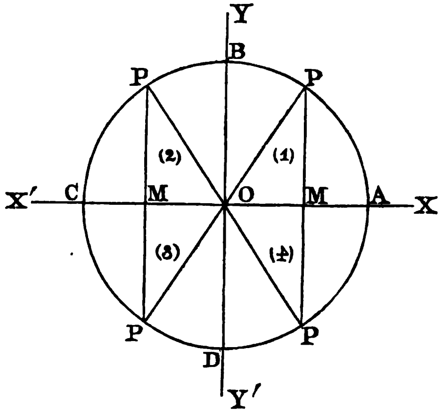
$$\tan(90^\circ + A) = \frac{\sin(90^\circ + A)}{\cos(90^\circ + A)} = \frac{\cos A}{-\sin A} = -\cot A,$$

$$\therefore \cot(90^\circ + A) = -\tan A.$$

$$\sec(90^\circ + A) = \frac{1}{\cos(90^\circ + A)} = \frac{1}{-\sin A} = -\operatorname{cosec} A,$$

$$\operatorname{cosec}(90^\circ + A) = \frac{1}{\sin(90^\circ + A)} = \frac{1}{\cos A} = \sec A.$$

*17. *To trace the changes in the sine and cosine of an angle as it varies from* 0° *to* 360°.



Let YY' cut the initial line $X'X$ at right angles in O, and suppose a line OP of constant length to revolve from its initial position OA in OX, so that P describes the circle $ABCD$; then from the right-angled triangle POM in any of the four quadrants we have $\sin POX = \frac{PM}{OP}$.

When OP coincides with OA, PM vanishes, and thus $\sin 0° = 0$.

As OP moves through the first quadrant (1), PM is positive and continually increases until OP coincides with OB, and then PM also coincides with OB or OP; thus as the angle POX increases from 0° to 90°, $\sin POX$ increases from 0 to 1.

As OP moves through the second quadrant (2), PM is positive and continually decreases until OP coincides with OC, and then PM vanishes; thus as the angle POX, swept out by OP, increases from 90° to 180°, $\sin POX$ decreases from 1 to 0.

As OP moves through the third quadrant (3), PM is *negative* and increases *numerically* until OP coincides with OD; thus as the angle POX increases from 180° to 270°, its sine is *negative* and increases *numerically* from 0 to -1.

As OP moves through the fourth quadrant (4), PM is *negative* and decreases *numerically* until OP coincides again with OA; thus as the angle POX increases from 270° to 360°, its sine is *negative* and decreases *numerically* from -1 to 0.

We thus see that $\sin 0° = 0$, $\sin 90° = 1$, $\sin 180° = 0$, $\sin 270° = -1$, $\sin 360° = 0$;

or, in circular measure (see Art. 7),

$$\sin 0 = 0,\ \sin\frac{\pi}{2} = 1,\ \sin\pi = 0,\ \sin\frac{3\pi}{2} = -1,\ \sin 2\pi = 0.$$

Again, $\cos POX = \dfrac{OM}{OP}$.

When OP coincides with OA, then $OM = OA$ or OP, and thus $\cos 0° = 1$.

As OP moves through the first quadrant, OM is positive and continually decreases until OP coincides with OB, when

OM vanishes; thus as POX increases from 0° to 90°, its cosine decreases from 1 to 0.

As OP moves through the second quadrant, OM is *negative* and increases *numerically* until OP coincides with OC; thus as the angle POX increases from 90° to 180°, its cosine is *negative* and increases *numerically* from 0 to -1.

In the same way it is seen that, as POX increases from 180° to 270°, its cosine is *negative* and decreases *numerically* from -1 to 0, and as POX increases from 270° to 360°, its cosine is positive and increases from 0 to 1.

Thus we see that $\cos 0° = 1$, $\cos 90° = 0$, $\cos 180° = -1$, $\cos 270° = 0$, $\cos 360° = 1$,

or, in circular measure,

$$\cos 0 = 1, \quad \cos\frac{\pi}{2} = 0, \quad \cos\pi = -1, \quad \cos\frac{3\pi}{2} = 0, \quad \cos 2\pi = 1.$$

*18. *To trace the changes in the tangent of an angle as it varies from* 0° *to* 360°.

$\tan POX = \dfrac{PM}{OM}$. (See fig. Art. 17.)

When OP coincides with OA, then $PM = 0$, and thus $\tan 0° = 0$.

As OP moves through the first quadrant, PM and OM are both positive, PM continually increases and OM continually decreases until OP coincides with OB; thus, as POX increases from 0° to 90°, its tangent is positive and increases from 0 without limit; so that by taking POX sufficiently near to 90°, we can make $\tan POX$ as great as we please, or, more briefly, $\tan 90° = \infty$.

As OP moves through the second quadrant, PM is positive and continually decreases, and OM is *negative* and increases *numerically* until OP coincides with OC; thus, as the angle POX increases from 90° to 180°, its tangent is *negative* and decreases *numerically* from $-\infty$ to 0.

As OP moves through the third quadrant, both PM and and OM are *negative*, and therefore the tangent is always *positive*; also PM increases and OM decreases *numerically* until OP coincides with OD; thus, as the angle POX increases from 180° to 270°, its tangent increases from 0 to ∞.

As OP moves through the fourth quadrant, PM is *negative* and decreases *numerically*, and OM is *positive* and increases; thus, as the angle POX increases from 270° to 360°, its tangent is always *negative* and decreases *numerically* from $-\infty$ to 0.

We thus see that $\tan 0° = 0$, $\tan 90° = \infty$, $\tan 180° = 0$, $\tan 270° = \infty$, $\tan 360° = 0$,

or, in the circular measure,

$$\tan 0 = 0,\ \tan\frac{\pi}{2} = \infty,\ \tan\pi = 0,\ \tan\frac{3\pi}{2} = \infty,\ \tan 2\pi = 0.$$

N.B. When the angle POX is in the first quadrant and indefinitely near to 90°, then $\tan POX$ is indefinitely great and positive; when POX is in the second quadrant and indefinitely near to 90°, then $\tan POX$ is indefinitely great and *negative*.

Thus $\tan 90° = +\infty$ (OP in the first quadrant),

$\tan 90° = -\infty$ (OP in the second quadrant),

$\tan 270° = +\infty$ (OP in the third quadrant),

$\tan 270° = -\infty$ (OP in the fourth quadrant).

19. Since (Art. 11)

$$\sec A = \frac{1}{\cos A},\quad \operatorname{cosec} A = \frac{1}{\sin A},\quad \text{and } \cot A = \frac{1}{\tan A}.$$

We may deduce the changes in the secant, cosecant, and cotangent of an angle, as it varies from 0° to 360°, by means of the known changes in the cosine, sine, and tangent, already traced. It will thus appear that the secant and cosecant may have any value between -1 and $-\infty$ and between $+1$ and $+\infty$, that the tangent and cotangent may have any value between $-\infty$ and $+\infty$, and that the sine and cosine may have any value between -1 and $+1$.

Since, $\text{vers}A = 1 - \cos A$ (Art. 10),

and $\text{covers}A = 1 - \sin A$,

therefore $\text{vers}A$ and $\text{covers}A$ are always *positive*, and may have any value between 0 and 2.

20. We see from the last three Articles that the sine and cosine change their signs whenever they pass through the value zero, and that the tangent, cotangent, secant, and cosecant change their signs whenever they pass through the value infinity, and that they do not change their signs in passing through any other values.

21. *To show that the Trigonometrical Ratios of any angle whatever may be expressed in terms of the Trigonometrical Ratios of some positive angle not exceeding a right angle.*

We need only consider the case of the sine and cosine, since the other Trigonometrical Ratios can be expressed directly in terms of these.

By Art. (12), $\sin(-A) = -\sin A$,

and $\cos(-A) = \cos A$,

therefore we can make the sine and cosine of any *negative* angle depend upon those of the *corresponding positive* angle.

By Art. (10) any multiple of four right angles may be taken away from, or added to, an angle without causing any change in its Trigonometrical Ratios, and thus we may replace any angle by an angle less than four right angles.

By Art. (14) $\sin(180° + A) = -\sin A$,
and $\cos(180° + A) = -\cos A$,

and thus we may make the sine and cosine of any angle depend upon those of an angle *not greater than* 180°.

By Art. (13) $\sin(180° - A) = \sin A$,
and $\cos(180° - A) = -\cos A$,

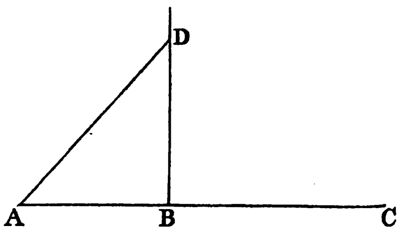
and thus we may make the sine and cosine of any angle depend upon those of an angle *not greater than* 90°.

*CHAPTER III.

CONSTRUCTION OF ANGLES WITH ASSIGNED TRIGONOMETRICAL RATIOS. GENERAL FORMULAE FOR ANGLES WITH GIVEN TRIGONOMETRICAL RATIOS.

22. *To construct an angle with a given* positive *sine or cosine.*

Let $AB:BC$ or $\frac{AB}{BC}$ be the given sine or cosine, BC being $> AB$.



Draw the indefinite straight line BD perpendicular to AB, and inflect $AD = BC$ (that is, from the centre A at a distance equal to BC describe a circle cutting BD in D, and join AD).

Then $\cos A = \frac{AB}{AD} = \frac{AB}{BC}$,

and $\sin D = \sin(90° - A) = \cos A$, by Art. (15), $= \frac{AB}{BC}$.

Hence, if the sine be given $= \frac{AB}{BC}$, D is the required angle, and if the cosine be given $= \frac{AB}{BC}$, A is the required angle.

23. *To construct an angle with a given* positive *tangent or cotangent.*

(See fig. Art. 22.)

Let $AB:BC$ or $\frac{AB}{BC}$ be the given tangent or cotangent.

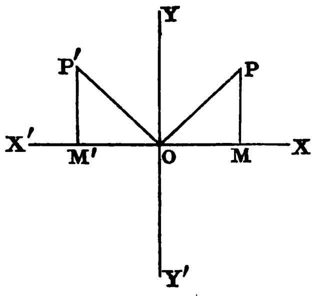
Draw BD perpendicular to AB, and $= BC$, and join AD.

Then $\cot A = \frac{AB}{BD} = \frac{AB}{BC}$,

and $\tan D = \cot A = \frac{AB}{BD} = \frac{AB}{BC}$.

Hence, if the cotangent be given $= \frac{AB}{BC}$, A is the required angle; and if the tangent be given $= \frac{AB}{BC}$, D is the required angle.

24. *To find a general expression for all the angles which have a given sine.*



Let POM be the least positive angle which has the given sine, and let α be its circular measure. Make the angle $P'OM' = POM$, so that the circular measure of the angle $P'OM$ is $\pi - \alpha$.

It is evident from the figure that all the angles terminated by OP and OP' have the given sine (see Art. 13).

Therefore (by Art. 10), if m be zero or a positive or negative integer, the formula $2m\pi + \alpha$ will comprehend all the angles terminated by OP and $2m\pi + \pi - \alpha$, or $(2m+1)\pi - \alpha$ will comprehend all the angles terminated by OP'.

Now $n\pi + (-1)^n . \alpha$, where n is zero or any positive or negative integer, will embrace both the formulae $2m\pi + \alpha$ and

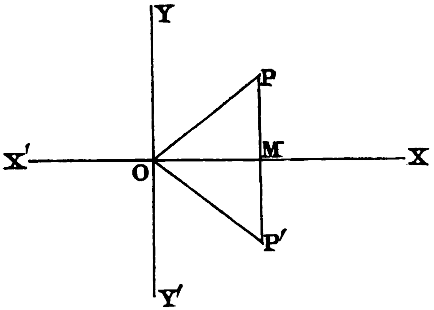
$(2m+1)\pi-\alpha$; for if n be an even integer, positive or negative, and $=2m$, $n\pi+(-1)^n.\alpha$ becomes $2m\pi+\alpha$, and if n be an odd integer, positive or negative, and $=2m+1$, $n\pi+(-1)^n.\alpha$ becomes $(2m+1)\pi-\alpha$.

Thus the formula $n\pi+(-1)^n.\alpha$ includes *all* the angles, and *no others*, in $2m\pi+\alpha$ and $(2m+1)\pi-\alpha$, that is, all the angles which have the same sine as α.

$\therefore \sin\{n\pi+(-1)^n.\alpha\}=\sin\alpha$.

The formula $n\pi+(-1)^n.\alpha$ also includes all the angles which have the same cosecant or coversed sine as α.

25. *To find a general expression for all the angles which have a given cosine.*



Let POM be the least positive angle which has the given cosine, and let α be its circular measure.

Make the $\angle P'OM=POM$, so that the circular measure of $P'OM$ is $-\alpha$.

It is evident from the figure that all the angles terminated by OP and OP' have the given cosine. (See Art. 12.)

Therefore (by Art. 10), if n be zero or any positive or negative integer, the formula $2n\pi+\alpha$ comprehends all the angles terminated by OP and $2n\pi-\alpha$ comprehends all the angles terminated by OP'.

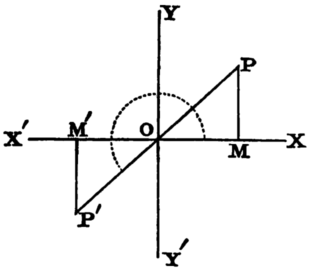
Thus the formula $2n\pi \pm \alpha$ includes *all* the angles, and *no others*, which have the same cosine as α.

$\therefore \cos(2n\pi \pm \alpha) = \cos\alpha$.

The formula $2n\pi \pm \alpha$ also includes all the angles which have the same secant or versed sine as α.

26. *To find a general expression for all the angles which have a given tangent.*

Let POM be the least positive angle which has the given tangent, and let α be its circular measure.



Produce PO to P', so that the circular measure of the angle $P'OM$, subtended at O by the dotted arc, is $\pi + \alpha$.

It is evident from the figure that all the angles terminated by OP and OP' have the given tangent. (See Art. 14.)

Therefore (by Art. 10), if m be zero or any positive or negative integer, the formula $2m\pi + \alpha$ comprehends all the angles terminated by OP, and $2m\pi + \pi + \alpha$ or $(2m+1)\pi + \alpha$ comprehends all the angles terminated by OP'.

The formula $n\pi + \alpha$ evidently comprehends *all* the angles, and *no others*, included in these two formulae, that is, all the angles which have the same tangent as α.

$\therefore \tan(n\pi + \alpha) = \tan\alpha$.

The formula $n\pi + \alpha$ also includes all the angles which have the same cotangent as α.

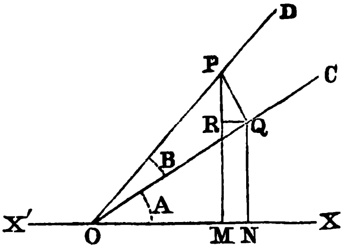
CHAPTER IV.

TRIGONOMETRICAL FUNCTIONS OF TWO ANGLES.

27. *To find the sine and cosine of* A + B *in terms of the sines and cosines of* A *and* B.

Let the angle $XOC = A$ and $COD = B$; then the angle $XOD = A + B$.

From P, any point in OD, draw PM perpendicular to OX and PQ perpendicular to OC; draw QN perpendicular to OX and QR perpendicular to PM.



Then the angle $QPR = 90° - PQR = RQO = A$.

Now $\sin(A + B) = \dfrac{PM}{OP} = \dfrac{RM + PR}{OP} = \dfrac{QN + PR}{OP}$,

$$= \frac{QN}{OQ} \cdot \frac{OQ}{OP} + \frac{PR}{PQ} \cdot \frac{PQ}{OP},$$

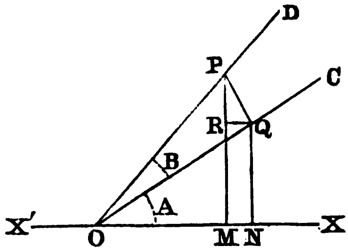
$$= \sin A \cos B + \cos A \sin B.$$

$\cos(A + B) = \dfrac{OM}{OP} = \dfrac{ON - MN}{OP} = \dfrac{ON - QR}{OP}$,

$$= \frac{ON}{OQ} \cdot \frac{OQ}{OP} - \frac{QR}{PQ} \cdot \frac{PQ}{OP},$$

$$= \cos A \cos B - \sin A \sin B.$$

*28. *To find the tangent of* A + B *in terms of the tangents of* A *and* B.



$$\tan(A+B)=\frac{PM}{OM}=\frac{RM+PR}{ON-MN}=\frac{QN+PR}{ON-QR},$$

$$=\frac{\frac{QN}{ON}+\frac{PR}{ON}}{1-\frac{QR}{ON}}.$$

$$=\frac{\frac{QN}{ON}+\frac{PR}{ON}}{1-\frac{QR}{QN}\cdot\frac{QN}{ON}}.$$

Now from the similar triangles PQR, QON, we have

$$PR:PQ::ON:OQ,\ \therefore\ \frac{PR}{ON}=\frac{PQ}{OQ}=\tan B,$$

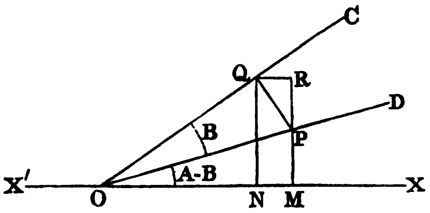
and $PQ:QR::OQ:QN$, $\therefore\ \frac{QR}{QN}=\frac{PQ}{OQ}=\tan B.$

Also $\frac{QN}{ON}=\tan A.$

$$\therefore\ \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

29. *To find the sine and cosine of* A − B *in terms of the sines and cosines of* A *and* B.

Let the angle $XOC=A$ and $COD=B$; then the angle $XOD=A-B$.



From P, any point in OD, draw PM perpendicular to OX, and PQ perpendicular to OC; draw QN perpendicular to OX, and QR perpendicular to PM produced.

Then the angle $QPR = 90° - PQR = RQC = A$.

$$\text{Now } \sin(A-B) = \frac{PM}{OP} = \frac{RM-PR}{OP} = \frac{QN-PR}{OP},$$

$$= \frac{QN}{OQ} \cdot \frac{OQ}{OP} - \frac{PR}{PQ} \cdot \frac{PQ}{OP},$$

$$= \sin A \cos B - \cos A \sin B.$$

$$\cos(A-B) = \frac{OM}{OP} = \frac{ON+MN}{OP} = \frac{ON+QR}{OP},$$

$$= \frac{ON}{OQ} \cdot \frac{OQ}{OP} + \frac{QR}{PQ} \cdot \frac{PQ}{OP},$$

$$= \cos A \cos B + \sin A \sin B.$$

*30. *To find the tangent of* A − B *in terms of the tangents of* A *and* B.

(See fig. Art. 29.)

$$\tan(A-B) = \frac{PM}{OM} = \frac{RM-PR}{ON+MN},$$

$$= \frac{QN-PR}{ON+QR},$$

$$= \frac{\frac{QN}{ON} - \frac{PR}{ON}}{1 + \frac{QR}{ON}}.$$

$$= \frac{\frac{QN}{ON} - \frac{PR}{ON}}{1 + \frac{QR}{QN} \cdot \frac{QN}{ON}}.$$

Now from the similar triangles PQR, QON, we have

$$PR : PQ :: ON : OQ, \therefore \frac{PR}{ON} = \frac{PQ}{OQ} = \tan B,$$

and $$PQ : QR :: OQ : QN, \therefore \frac{QR}{QN} = \frac{PQ}{OQ} = \tan B.$$

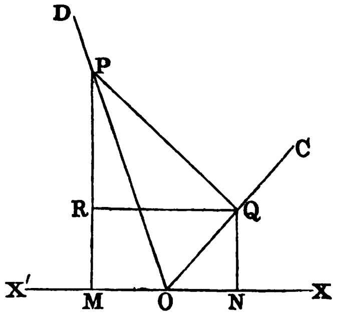
Also $\dfrac{QN}{ON} = \tan A$,

$$\therefore \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

31. The formulae in the four preceding Articles have only been proved for positive values of A, B, $A+B$ and $A-B$, less than 90°. In the next Article, however, they will be shewn to hold universally for all angles. *We will now prove them when* A *and* B *are each less than* 90°, *but* $A + B$ *greater than* 90°.

Let the angle $XOC = A$, and $COD = B$; then the angle $XOD = A + B$.

From P, any point in OD, draw PM perpendicular to $X'X$, and PQ perpendicular to OC; draw QN perpendicular to OX, and QR perpendicular to PM.



Then the angle $QPR = 90° - PQR = RQO = A$.

$$\text{Now } \sin(A + B) = \frac{PM}{OP} = \frac{RM + PR}{OP} = \frac{QN + PR}{OP},$$

$$= \frac{QN}{OQ} \cdot \frac{OQ}{OP} + \frac{PR}{PQ} \cdot \frac{PQ}{OP},$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\cos(A+B) = \frac{OM}{OP} = \frac{ON - MN}{OP}, \text{ since } OM \text{ is } \textit{negative},$$

$$= \frac{ON - QR}{OP},$$

$$= \frac{ON}{OQ} \cdot \frac{OQ}{OP} - \frac{QR}{PQ} \cdot \frac{PQ}{OP},$$

$$= \cos A \cos B - \sin A \sin B.$$

32. *To prove the following four formulae universally true for all values of* A *and* B.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \ldots\ldots(1),$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \ldots\ldots(2),$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \ldots\ldots(3),$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \ldots\ldots(4).$$

These formulae have been proved for values of A and B less than 90°, A being greater than B in (3) and (4).

The restriction of A being greater than B may be removed from (3) and (4); for let B be greater than A;

then, by Art. (12), $\sin(A-B) = -\sin(B-A)$,

$$\cos(A-B) = \cos(B-A).$$

Now since $B-A$ is positive and B and A each less than 90°, the formulae (3) and (4) hold for $\sin(B-A)$ and $\cos(B-A)$.

$\therefore \quad \sin(B-A) = \sin B \cos A - \cos B \sin A$,

and $\cos(B-A) = \cos B \cos A + \sin B \sin A$.

Hence $\sin(A-B) = \sin A \cos B - \cos A \sin B$,

and $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

Thus the four formulae hold for all values of A and B between 0° and 90°.

Again, these limits may be increased by 90° for either or both of the angles A and B.

For, by Art. (16),

$$\sin(90^\circ + A + B) = \cos(A + B),$$
$$= \cos A \cos B - \sin A \sin B \text{ by } (2),$$
$$= \sin(90^\circ + A)\cos B + \cos(90^\circ + A)\sin B,$$
$$\text{or } = \cos A \sin(90^\circ + B) + \sin A \cos(90^\circ + B).$$

Therefore if (2) hold for any values of A and B, (1) will hold when either angle has been increased by 90°, and thus (1) can be shewn to hold for any positive angles however large; and in the same way the three other formulae can be shewn to hold for any positive values of A and B however large.

Thus the four formulae hold for any positive values of A and B.

They also hold for any *negative* values of A and B; for if B be negative in (1), it is reduced to (3) with A and B positive, which case has already been established.

Suppose then A and B both negative, and let $A = -A'$ and $B = -B'$;

$$\text{then } \sin(A + B) = \sin(-A' - B') = -\sin(A' + B'),$$
$$= -\sin A' \cos B' - \cos A' \sin B', \text{ by what has been already shewn.}$$
$$= -\sin(-A)\cos(-B) - \cos(-A)\sin(-B),$$
$$= \sin A \cos B + \cos A \sin B.$$

Similarly the other formulae can be shewn to hold good when A and B are both negative, or when only one of these angles is negative.

33. From the formulae in Art. (32) we have

$$\sin(A + B) + \sin(A - B) = 2\sin A \cos B \ldots\ldots(1),$$
$$\sin(A + B) - \sin(A - B) = 2\cos A \sin B \ldots\ldots(2),$$
$$\cos(A - B) + \cos(A + B) = 2\cos A \cos B \ldots\ldots(3),$$
$$\cos(A - B) - \cos(A + B) = 2\sin A \sin B \ldots\ldots(4).$$

If we consider $A+B$ and $A-B$ as single angles, then A is half their sum and B half their difference.

We may therefore enunciate these four formulae as follows:

(1) *The sum of the sines of any two angles is equal to twice the sine of half their sum multiplied into the cosine of half their difference.*

(2) *The difference of the sines of any two angles is equal to twice the cosine of half their sum into the sine of half their difference.*

(3) *The sum of the cosines of any two angles is equal to twice the cosine of half their sum into the cosine of half their difference.*

(4) *The difference of the cosines of two angles is equal to twice the sine of half their sum into the sine of half their difference.*

It ought to be carefully noted that in the case of the difference of the cosines the *smaller angle* $A-B$ stands *first* on the left-hand side.

Hence

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$
$$\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B),$$
$$\cos B+\cos A=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$
$$\cos B-\cos A=2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

34. $\sin(A+B)\sin(A-B)$

$$=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B),$$
$$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B,$$
$$=\sin^2 A(1-\sin^2 B)-\cos^2 A\sin^2 B,$$
$$=\sin^2 A-\sin^2 B(\sin^2 A+\cos^2 A),$$
$$=\sin^2 A-\sin^2 B, \text{ since } \cos^2 A+\sin^2 A=1, \text{ by Art. (11).}$$

And $\cos(A+B)\cos(A-B)$

$$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B),$$
$$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B,$$
$$= \cos^2 A(1-\sin^2 B) - \sin^2 A\sin^2 B,$$
$$= \cos^2 A - \sin^2 B(\cos^2 A + \sin^2 A),$$
$$= \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

35. $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$

Dividing both numerator and denominator of this fraction by $\cos A\cos B$, we have

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \text{.........(1)},$$

$$\therefore \cot(A+B) = \frac{1 - \tan A\tan B}{\tan A + \tan B}.$$

Multiply the numerator and denominator of this fraction by $\cot A\cot B$, then since $\tan A\cot A = 1$, we have

$$\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A} \text{.........(2)}.$$

In (1) and (2) put $B = A$.

Then $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ ....(3)

and $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$ ....(4)

In (1) and (2) for B write $-B$, then since $\tan(-B) = -\tan B$ and $\cot(-B) = -\cot B$, we have

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \text{............(5)}$$

and $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$ ....(6).

36. $$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}\quad\text{(Art. 33.)}$$
$$=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)},$$
$$\frac{\cos B-\cos A}{\cos B+\cos A}=\frac{2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)},$$
$$=\tan\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B).$$

37. $$\sin 2A=\sin(A+A)=\sin A\cos A+\cos A\sin A,$$
$$=2\sin A\cos A,$$
$$\cos 2A=\cos(A+A)=\cos A\cos A-\sin A\sin A,$$
$$=\cos^2A-\sin^2A,$$
$$=2\cos^2A-1,$$
$$=1-2\sin^2A,$$
$\therefore\ 1+\cos 2A=2\cos^2A$;
and $1-\cos 2A=2\sin^2A$,
$$\therefore\ \frac{1-\cos 2A}{1+\cos 2A}=\frac{2\sin^2A}{2\cos^2A}=\tan^2A.$$
Also $\sin 2A=\dfrac{2\sin A\cos A}{\cos^2A+\sin^2A}$, since $\cos^2A+\sin^2A=1$,

$=\dfrac{2\tan A}{1+\tan^2A}$, by dividing both numerator and denominator of the last fraction by $\cos^2A$.
$$\cos 2A=\frac{\cos^2A-\sin^2A}{\cos^2A+\sin^2A},$$
$$=\frac{1-\tan^2A}{1+\tan^2A}.$$
$$\frac{1+\cos 2A}{\sin 2A}=\frac{2\cos^2A}{2\sin A\cos A}=\cot A,$$
or $\operatorname{cosec}2A+\cot 2A=\cot A$,

$\therefore\ \operatorname{cosec}2A=\cot A-\cot 2A.$
$$\frac{1-\cos 2A}{\sin 2A}=\frac{2\sin^2A}{2\sin A\cos A}=\tan A,$$
or $\operatorname{cosec}2A-\cot 2A=\tan A.$

38. $\cot A + \tan A = \dfrac{\cos A}{\sin A} + \dfrac{\sin A}{\cos A} = \dfrac{\cos^2 A + \sin^2 A}{\sin A \cos A}$

$$= \frac{1}{\sin A \cos A} = \frac{2}{\sin 2A} = 2 \operatorname{cosec} 2A\,;$$

$$\cot A - \tan A = \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A} = \frac{\cos^2 A - \sin^2 A}{\sin A \cos A}$$

$$= \frac{\cos 2A}{\sin A \cos A} = \frac{2\cos 2A}{\sin 2A} = 2\cot 2A.$$

39. $\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A$

$$= 2\sin A \cos^2 A + (1 - 2\sin^2 A)\sin A$$

$$= 3\sin A - 4\sin^3 A\,;$$

$$\cos 3A = \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$$

$$= (2\cos^2 A - 1)\cos A - 2\cos A \sin^2 A$$

$$= (2\cos^2 A - 1)\cos A - 2\cos A\,(1 - \cos^2 A)$$

$$= 4\cos^3 A - 3\cos A.$$

Hence $\tan 3A = \dfrac{\sin 3A}{\cos 3A} = \dfrac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A}$

$$= \frac{3\sin A(\cos^2 A + \sin^2 A) - 4\sin^3 A}{4\cos^3 A - 3\cos A(\cos^2 A + \sin^2 A)}$$

$$= \frac{3\sin A \cos^2 A - \sin^3 A}{\cos^3 A - 3\cos A \sin^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A},$$

by dividing the numerator and denominator of the last fraction by $\cos^3 A$.

40. Let $A + B + C = 180°$,

$\therefore\ A = 180° - (B + C)$,

$\therefore\ \cos A = -\cos(B + C) = -\cos B \cos C + \sin B \sin C$,

$\therefore\ (\cos A + \cos B \cos C)^2 = \sin^2 B \sin^2 C = (1 - \cos^2 B)(1 - \cos^2 C)$,

$\therefore\ \cos^2 A + 2\cos A \cos B \cos C + \cos^2 B \cos^2 C = 1 - \cos^2 B - \cos^2 C + \cos^2 B \cos^2 C$,

$\therefore\ \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1 \ldots\ldots (1).$

Again, $A+B=180^\circ - C$,

$\therefore \tan(A+B) = -\tan C$,

or $\dfrac{\tan A+\tan B}{1-\tan A\tan B} = -\tan C$,

$\therefore \tan A+\tan B = -\tan C(1-\tan A\tan B)$,

$\therefore \tan A+\tan B+\tan C = \tan A\tan B\tan C$......(2).

41. Let $\alpha+\beta+\gamma=2\pi$,

$\therefore \alpha = 2\pi-(\beta+\gamma)$,

$\therefore \cos\alpha = \cos(\beta+\gamma) = \cos\beta\cos\gamma - \sin\beta\sin\gamma$,

$\therefore (\cos\alpha-\cos\beta\cos\gamma)^2 = \sin^2\beta\sin^2\gamma$,

$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma=1$.

Also $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$. (See last Article.)

*CHAPTER V.

TRIGONOMETRICAL RATIOS OF KNOWN ANGLES. FORMULAE OF VERIFICATION AND FOR THE DIVISION OF ANGLES.

42. *Find the principal Trigonometrical Ratios of the following angles:* 30°, 60°, 45°, 15°, 75°, 36°, 54°, 18°, 72°, 9°, 81°, 27°, *and* 63°.

The Trigonometrical Ratios of all these angles will be positive, since the angles are positive and less than 90°.

$$\sin 60° = 2\sin 30° \cos 30° \quad \text{(Art. 37)}$$

$$= 2\sin 30° \sin 60°, \text{ since } \cos 30° = \sin(90° - 30°) \quad \text{(Art. 15)};$$

$$\therefore\ 1 = 2\sin 30°, \ \therefore\ \sin 30° = \cos 60° = \tfrac{1}{2},$$

$$\therefore\ \cos 30° = \sin 60° = \sqrt{(1 - \sin^2 30°)} = \sqrt{(1 - \tfrac{1}{4})} = \tfrac{1}{2}\sqrt{3},$$

$$\therefore\ \tan 60° = \frac{\sin 60°}{\cos 60°} = \frac{\tfrac{1}{2}\sqrt{3}}{\tfrac{1}{2}} = \sqrt{3} = \cot 30°,$$

$$\therefore\ \cot 60° = \frac{1}{\sqrt{3}} = \tan 30°.$$

$$\sin^2 45° + \cos^2 45° = 1, \text{ and } \sin 45° = \cos 45°,$$

$$\therefore\ 2\cos^2 45° = 1,$$

$$\therefore\ \cos 45° = \sin 45° = \frac{1}{\sqrt{2}},$$

$$\therefore\ \tan 45° = \cot 45° = 1.$$

$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$
$$= \frac{1}{\sqrt{2}} \cdot \tfrac{1}{2}\sqrt{3} + \frac{1}{\sqrt{2}} \cdot \tfrac{1}{2}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 15^\circ.$$

$$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}} = \sin 15^\circ,$$

$$\therefore \tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = 2 + \sqrt{3} = \cot 15^\circ,$$

$$\therefore \cot 75^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = 2 - \sqrt{3} = \tan 15^\circ.$$

Let $A = 18^\circ$, $\therefore 3A = 54^\circ$, $2A = 36^\circ$, and $3A + 2A = 90^\circ$,

$\therefore \cos 3A = \sin 2A$,

$\therefore 4\cos^3 A - 3\cos A = 2\sin A \cos A$ (Arts. 39, 37),

$\therefore 4\cos^2 A - 3 = 2\sin A$,

or $4(1 - \sin^2 A) - 3 = 2\sin A$,

$\therefore 4\sin^2 A + 2\sin A = 1$,

$$\therefore \sin A = \frac{-1 \pm \sqrt{5}}{4}.$$

Since $\sin 18^\circ$ is positive we must take the positive root;

$\therefore \sin 18^\circ = \cos 72^\circ = \frac{1}{4}(\sqrt{5} - 1)$,

$$\therefore \cos 18^\circ = \sin 72^\circ = \sqrt{(1 - \sin^2 18^\circ)} = \sqrt{\left(1 - \frac{6 - 2\sqrt{5}}{16}\right)}$$
$$= \tfrac{1}{4}\sqrt{(10 + 2\sqrt{5})},$$

$\therefore \cos 36^\circ = 2\cos^2 18^\circ - 1$ (Art. 37)

$$= \frac{10 + 2\sqrt{5}}{8} - 1 = \tfrac{1}{4}(\sqrt{5} + 1) = \sin 54^\circ,$$

$$\therefore \sin 36^\circ = \sqrt{(1 - \cos^2 36^\circ)} = \sqrt{\left(1 - \frac{6 + 2\sqrt{5}}{16}\right)}$$
$$= \tfrac{1}{4}\sqrt{(10 - 2\sqrt{5})} = \cos 54^\circ.$$

By Art. (33) we have

$$\sin 54^\circ + \sin 36^\circ = 2\sin 45^\circ \cos 9^\circ = \sqrt{2} \,.\, \cos 9^\circ,$$

$$\sin 54^\circ - \sin 36^\circ = 2\cos 45^\circ \sin 9^\circ = \sqrt{2} \,.\, \sin 9^\circ,$$

$$\therefore\ \cos 9^\circ = \sin 81^\circ = \frac{\sin 54^\circ + \sin 36^\circ}{\sqrt{2}} = \frac{\sqrt{5}+1+\sqrt{(10-2\sqrt{5})}}{4\sqrt{2}}$$

$$= \frac{\sqrt{(3+\sqrt{5})} + \sqrt{(5-\sqrt{5})}}{4},$$

since $\sqrt{(3+\sqrt{5})} = \frac{\sqrt{5}+1}{\sqrt{2}}$, by the ordinary rule for extracting the square root of a Binomial Surd.

And

$$\sin 9^\circ = \cos 81^\circ = \frac{\sqrt{5}+1-\sqrt{(10-2\sqrt{5})}}{4\sqrt{2}} = \frac{\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}}{4}.$$

By Art. (33) we have

$$\sin 63^\circ + \sin 27^\circ = 2\sin 45^\circ \cos 18^\circ = \frac{2\cos 18^\circ}{\sqrt{2}},$$

$$\sin 63^\circ - \sin 27^\circ = 2\cos 45^\circ \sin 18^\circ = \frac{2\sin 18^\circ}{\sqrt{2}},$$

$$\therefore\ \sin 63^\circ = \cos 27^\circ = \frac{\cos 18^\circ + \sin 18^\circ}{\sqrt{2}} = \frac{\sqrt{(10+2\sqrt{5})} + \sqrt{5} - 1}{4\sqrt{2}},$$

$$\sin 27^\circ = \cos 63^\circ = \frac{\cos 18^\circ - \sin 18^\circ}{\sqrt{2}} = \frac{\sqrt{(10+2\sqrt{5})} - (\sqrt{5}-1)}{4\sqrt{2}}.$$

The Trigonometrical Ratios of 30° and 60° may also be easily found by drawing a perpendicular from one angle of an equilateral triangle to the opposite side.

42. *Reduce $\sin\theta + \cos\theta$ to a single term.*

$$\sin\theta + \cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + \cos\theta = 2\cos\frac{\pi}{4}\cos\left(\frac{\pi}{4} - \theta\right) \quad \text{(Art. 33)}$$

$$= \sqrt{2} \,.\, \cos\left(\frac{\pi}{4} - \theta\right).$$

43. By Art. (33) we have

$$\sin(36^\circ+A)-\sin(36^\circ-A)=2\cos 36^\circ \sin A=\tfrac{1}{2}(\surd 5+1)\sin A,$$

$$\sin(72^\circ+A)-\sin(72^\circ-A)=2\cos 72^\circ \sin A=\tfrac{1}{2}(\surd 5-1)\sin A,$$

$$\therefore\ \sin(36^\circ+A)+\sin(72^\circ-A)-\sin(36^\circ-A)-\sin(72^\circ+A)$$

$$=\tfrac{1}{2}(\surd 5+1)\sin A-\tfrac{1}{2}(\surd 5-1)\sin A=\sin A.$$

(Euler's Formula.)

$$\sin(54^\circ+A)+\sin(54^\circ-A)=2\sin 54^\circ \cos A=\tfrac{1}{2}(\surd 5+1)\cos A,$$

$$\sin(18^\circ+A)+\sin(18^\circ-A)=2\sin 18^\circ \cos A=\tfrac{1}{2}(\surd 5-1)\cos A,$$

$$\therefore\ \sin(54^\circ+A)+\sin(54^\circ-A)-\sin(18^\circ+A)-\sin(18^\circ-A),$$

$$=\tfrac{1}{2}(\surd 5+1)\cos A-\tfrac{1}{2}(\surd 5-1)\cos A=\cos A.$$

(Legendre's Formula.)

$$\sin(60^\circ+A)-\sin(60^\circ-A)=2\cos 60\sin A=\sin A,$$

$$\sin(45^\circ+A)-\sin(45^\circ-A)=2\cos 45^\circ \sin A=\surd 2\,.\sin A.$$

Euler's and Legendre's formulae are called *formulae of verification*, because they are used in verifying results in the computation of Trigonometrical Tables.

44. *Given* $\sin A$ *to find* $\sin\frac{A}{2}$ *and* $\cos\frac{A}{2}$.

$$2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A=\sin A, \text{ (Art. 37)},$$

$$\sin^2\tfrac{1}{2}A+\cos^2\tfrac{1}{2}A=1,$$

$$\therefore\ (\sin\tfrac{1}{2}A+\cos\tfrac{1}{2}A)^2=1+\sin A,$$

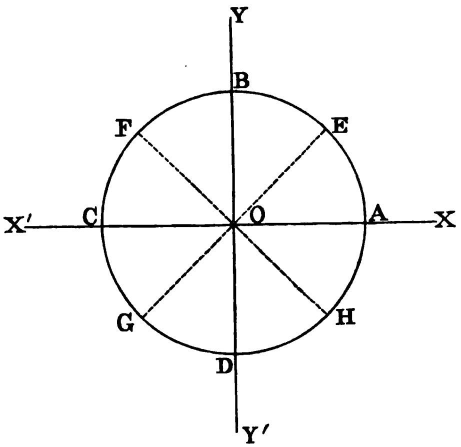
$$(\sin\tfrac{1}{2}A-\cos\tfrac{1}{2}A)^2=1-\sin A;$$

$$\therefore\ \left.\begin{array}{l}\sin\tfrac{1}{2}A+\cos\tfrac{1}{2}A=\pm\surd(1+\sin A)\\ \sin\tfrac{1}{2}A-\cos\tfrac{1}{2}A=\pm\surd(1-\sin A)\end{array}\right\}\ldots(1).$$

$$\therefore\ \left.\begin{array}{l}2\sin\tfrac{1}{2}A=\pm\surd(1+\sin A)\pm\surd(1-\sin A)\\ 2\cos\tfrac{1}{2}A=\pm\surd(1+\sin A)\mp\surd(1-\sin A)\end{array}\right\}\ldots(2).$$

The ambiguity of sign before the radicals in (1) and (2) may be removed by the particular value or limits assigned to A, as will appear from the next Article.

45. *To trace the change in the sign of* $\sin\frac{1}{2}A + \cos\frac{1}{2}A$ *and of* $\sin\frac{1}{2}A - \cos\frac{1}{2}A$ *as* $\frac{1}{2}A$ *varies from* 0° *to* 360°, *that is, as* A *varies from* 0° *to* 720°.



Let the *dotted* lines bisect the four quadrants.

It is clear that the sines and cosines of the angles terminated by the *dotted* lines are *numerically* equal, and that the *numerically greater* of the two ratios $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ will determine the signs of both expressions $\sin\frac{1}{2}A + \cos\frac{1}{2}A$ and $\sin\frac{1}{2}A - \cos\frac{1}{2}A$.

Now for values of $\frac{1}{2}A$ from $-45°$ to $+45°$, that is, through the angle HOE,

$\cos\frac{1}{2}A$ is greater than $\sin\frac{1}{2}A$;

therefore for those limits of $\frac{1}{2}A$

$\sin\frac{1}{2}A + \cos\frac{1}{2}A$ is positive,

and $\sin\frac{1}{2}A - \cos\frac{1}{2}A$ is negative;

$$\therefore \quad \sin\tfrac{1}{2}A + \cos\tfrac{1}{2}A = +\sqrt{(1 + \sin A)},$$

$$\sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A = -\sqrt{(1 - \sin A)}.$$

For values of $\frac{1}{2}A$ from 45° to 135°, that is, through the angle *EOF*,

$\sin\frac{1}{2}A$ is positive and greater than $\cos\frac{1}{2}A$,

$$\therefore \sin\tfrac{1}{2}A + \cos\tfrac{1}{2}A = +\sqrt{(1+\sin A)},$$
$$\sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A = +\sqrt{(1-\sin A)}.$$

For values of $\frac{1}{2}A$ from 135° to 225°, that is, through the angle *FOG*,

$\cos\frac{1}{2}A$ is negative and *numerically greater* than $\sin\frac{1}{2}A$,

$$\therefore \sin\tfrac{1}{2}A + \cos\tfrac{1}{2}A = -\sqrt{(1+\sin A)},$$
$$\sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A = +\sqrt{(1-\sin A)}.$$

For values of $\frac{1}{2}A$ from 225° to 315°, that is, through the angle *GOH*,

$\sin\frac{1}{2}A$ is negative and *numerically greater* than $\cos\frac{1}{2}A$,

$$\therefore \sin\tfrac{1}{2}A + \cos\tfrac{1}{2}A = -\sqrt{(1+\sin A)},$$
$$\sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A = -\sqrt{(1-\sin A)}.$$

The limits $-45°$ to $+45°$ of $\frac{1}{2}A$ include the angle *HOE*; these limits are therefore equivalent to the limits from 315° to 360°, and from 0° to 45°.

Hence we may arrange the above results as follows:

For values of $\frac{1}{2}A$ from

$$\begin{matrix}315° \text{ to } 360° \\ \text{and } 0° \text{ to } 45°\end{matrix}\left\{\begin{aligned}2\sin\tfrac{1}{2}A &= +\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)} \\ 2\cos\tfrac{1}{2}A &= +\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)}\end{aligned}\right.,$$

$$45° \text{ to } 135°\left\{\begin{aligned}2\sin\tfrac{1}{2}A &= +\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)} \\ 2\cos\tfrac{1}{2}A &= +\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)}\end{aligned}\right.,$$

$$135° \text{ to } 225°\left\{\begin{aligned}2\sin\tfrac{1}{2}A &= -\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)} \\ 2\cos\tfrac{1}{2}A &= -\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)}\end{aligned}\right.,$$

$$225° \text{ to } 315°\left\{\begin{aligned}2\sin\tfrac{1}{2}A &= -\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)} \\ 2\cos\tfrac{1}{2}A &= -\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)}\end{aligned}\right..$$

EXAMPLE.—*Find the values of* $\sin 22°\frac{1}{2}$ *and* $\cos 22°\frac{1}{2}$.

It is clear that $\cos 22°\frac{1}{2}$ is greater than $\sin 22°\frac{1}{2}$.

$$\therefore\ \sin 22°\tfrac{1}{2} + \cos 22°\tfrac{1}{2} = +\sqrt{(1+\sin 45°)} = \sqrt{(1+\tfrac{1}{2}\sqrt{2})} = \frac{\sqrt{(2+\sqrt{2})}}{\sqrt{2}};$$

$$\sin 22°\tfrac{1}{2} - \cos 22°\tfrac{1}{2} = -\sqrt{(1-\sin 45°)} = -\frac{\sqrt{(2-\sqrt{2})}}{\sqrt{2}}.$$

$$\therefore\ \sin 22°\tfrac{1}{2} = \frac{\sqrt{(2+\sqrt{2})} - \sqrt{(2-\sqrt{2})}}{2\sqrt{2}},$$

$$\cos 22°\tfrac{1}{2} = \frac{\sqrt{(2+\sqrt{2})} + \sqrt{(2-\sqrt{2})}}{2\sqrt{2}}.$$

Here we might, at once, have written down the formulae opposite the limits 0° to 45°, viz.,

$$2\sin\tfrac{1}{2}A = +\sqrt{(1+\sin A)} - \sqrt{(1-\sin A)},$$
$$2\cos\tfrac{1}{2}A = +\sqrt{(1+\sin A)} + \sqrt{(1-\sin A)},$$

and then have substituted in them 45° for A.

46. *To divide a given angle* (2α) *into two parts whose tangents shall be in a given ratio.*

Let $\alpha+\theta$ and $\alpha-\theta$ be the two parts required, and $\frac{m}{n}$ the given ratio;

then $\dfrac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)} = \dfrac{m}{n}$;

$$\therefore\ \frac{\tan(\alpha+\theta)-\tan(\alpha-\theta)}{\tan(\alpha+\theta)+\tan(\alpha-\theta)} = \frac{m-n}{m+n},$$

$$\therefore\ \frac{\sin(\alpha+\theta)\cos(\alpha-\theta)-\cos(\alpha+\theta)\sin(\alpha-\theta)}{\sin(\alpha+\theta)\cos(\alpha-\theta)+\cos(\alpha+\theta)\sin(\alpha-\theta)} = \frac{m-n}{m+n};$$

or $\dfrac{\sin 2\theta}{\sin 2\alpha} = \dfrac{m-n}{m+n}$,

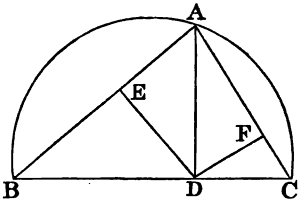
$\therefore\ \sin 2\theta = \dfrac{m-n}{m+n}.\sin 2\alpha$, which determines θ.

Hence $\alpha+\theta$ and $\alpha-\theta$ are known.

Geometrical Solution.

In the straight line BC take $BD:DC$ in the given ratio.

On BC describe the segment BAC containing the given angle, and draw DA perpendicular to BC. Join BA, AC.



Then the angles BAD, DAC are the required parts; for BAC is equal to the given angle, and

$$\tan BAD : \tan CAD :: \frac{BD}{AD} : \frac{DC}{AD} = BD : DC.$$

47. *To divide a given angle* (2α) *into two parts whose cosines shall be in a given ratio.*

Let $\alpha+\theta$ and $\alpha-\theta$ be the two parts required, and $\frac{m}{n}$ the given ratio;

then $\frac{\cos(\alpha-\theta)}{\cos(\alpha+\theta)}=\frac{m}{n}$,

$\therefore\ \frac{\cos(\alpha-\theta)-\cos(\alpha+\theta)}{\cos(\alpha-\theta)+\cos(\alpha+\theta)}=\frac{m-n}{m+n}$;

or $\tan\alpha\tan\theta=\frac{m-n}{m+n}$ (Art. 36),

$\therefore\ \tan\theta=\frac{m-n}{m+n}\cot\alpha$, which determines θ.

Hence $\alpha+\theta$ and $\alpha-\theta$ are known.

Geometrical Solution.

(See fig. Art 46.)

Make BAC equal to the given angle, and $AE:AF$ in the given ratio. Draw ED perpendicular to AB, and FD perpendicular to AC, meeting in D, and join AD; then BAD and DAC are the required parts; for

$$\cos BAD : \cos DAC :: \frac{AE}{AD} : \frac{AF}{AD} = AE : AF.$$

48. *To divide a given angle* (2α) *into two parts whose sines shall be in a given ratio.*

Let $\alpha+\theta$ and $\alpha-\theta$ be the two parts required, and $\frac{m}{n}$ the given ratio;

then $\dfrac{\sin(\alpha+\theta)}{\sin(\alpha-\theta)}=\dfrac{m}{n}$,

$\therefore\ \dfrac{\sin(\alpha+\theta)-\sin(\alpha-\theta)}{\sin(\alpha+\theta)+\sin(\alpha-\theta)}=\dfrac{m-n}{m+n}$;

or $\cot\alpha\tan\theta=\dfrac{m-n}{m+n}$ (Art. 36),

$\therefore\ \tan\theta=\dfrac{m-n}{m+n}\tan\alpha$, which determines θ.

Hence $\alpha+\theta$ and $\alpha-\theta$ are known.

Geometrical Solution.

(See fig. Art. 46.)

Make the angle EDF equal to the supplement of the given angle, and $DE:DF$ in the given ratio. Draw EA perpendicular to DE, and FA perpendicular to DF, meeting in A, and join AD; then BAD and DAC are the required angles; for BAC is equal to the given angle, and

$$\sin BAD:\sin DAC::\frac{DE}{AD}:\frac{DF}{AD}=DE:DF.$$

CHAPTER VI.

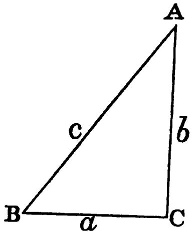
PROPERTIES OF TRIANGLES.

49. Let A, B, C denote the angles of any triangle, and a, b, c the sides respectively opposite to them.

When any three of these six quantities, except the three angles, are given, the remaining three can be determined, or the triangle can be constructed.

This can be easily shewn *Geometrically*, but it is the object of the present chapter to shew it *Trigonometrically* by solving the following PROBLEM. *Given any three parts of a triangle, except the three angles, to find the remaining three parts*, i.e. *to solve the triangle.*

50. *In a right-angled triangle either side is equal to the sine of the opposite angle, or cosine of the adjacent angle, multiplied into the hypotenuse.*



Let ABC be a triangle having a right angle at C; then, by definition,

$$\sin A = \cos B = \frac{a}{c},$$

$$\sin B = \cos A = \frac{b}{c}.$$

$$\therefore\ a = c\sin A = c\cos B,$$

$$b = c\sin B = c\cos A.$$

51. *In a right-angled triangle, either side is equal to the tangent of the opposite angle, or cotangent of the adjacent angle multiplied into the other side.*

(See fig. Art. 50.)

Let $C = 90°$; then, by definition,

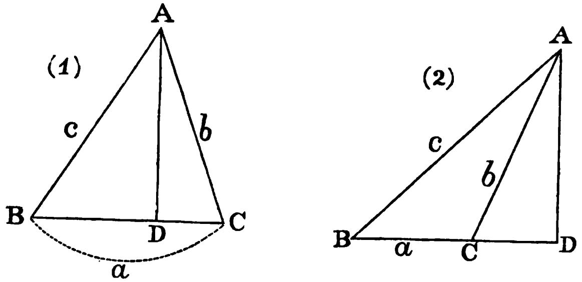
$$\tan A = \cot B = \frac{a}{b},$$

$$\tan B = \cot A = \frac{b}{a};$$

$\therefore\ a = b\tan A = b\cot B,$

$b = a\tan B = a\cot A.$

52. *In any triangle the sides are proportional to the sines of the opposite angles.*



Let ABC be any triangle, and from A draw AD perpendicular to BC or BC produced.

From the right-angled triangles ADB, ADC we have

$$AD = c\sin B,$$
$$AD = b\sin ACD = b\sin C,$$

since in (1) angle $ACD = C$, and in (2) $ACD = 180° - C$.

$\therefore\ c\sin B = b\sin C,\ \therefore\ \dfrac{c}{b} = \dfrac{\sin C}{\sin B}.$

If $C=90°$, we have (see fig. Art. 50)

$b=c\sin B$, $\therefore\ \frac{c}{b}=\frac{1}{\sin B}=\frac{\sin C}{\sin B}$, since $\sin C=\sin 90°=1$.

Therefore in every case $\frac{c}{b}=\frac{\sin C}{\sin B}$.

Similarly, $\frac{a}{c}=\frac{\sin A}{\sin C}$ and $\frac{a}{b}=\frac{\sin A}{\sin B}$.

Hence we have $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.

53. *In every triangle any side is equal to the sum of the products obtained by multiplying each of the other sides into the cosine of the angle between it and the first side.*

From fig. (1) of Art. 52, we have

$$a=BD+DC=c\cos B+b\cos C.$$

From fig. (2) of Art. 52,

$$a=BD-DC=c\cos B-b\cos(180°-C)$$
$$=c\cos B+b\cos C.$$

Similarly, $$b=a\cos C+c\cos A,$$
$$c=b\cos A+a\cos B.$$

54. *To express the cosine of an angle of a triangle in terms of the sides.*

See fig. (1) of Art. 52.

Let ABC be a triangle, having the angle C *acute;* then (Euclid II. 13)

$$c^2=a^2+b^2-2a\,.\,CD;$$

but $CD=b\cos C$, from the triangle ADC,

$\therefore\ c^2=a^2+b^2-2ab\cos C$.

See fig. (2) of Art 52.

Next, let C be *obtuse;* then (Euclid II. 12)

$$c^2 = a^2 + b^2 + 2a \,.\, CD;$$

but $CD = b\cos(180° - C) = -b\cos C$, from the triangle ADC,

$\therefore\ c^2 = a^2 + b^2 - 2ab\cos C$, which is the same as before.

Lastly, let $C = 90°$; then $\cos C = \cos 90° = 0$, and $c^2 = a^2 + b^2 - 2ab\cos C$ reduces to $c^2 = a^2 + b^2$, which we know to be true (Euclid I. 47).

Therefore in all cases we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Similarly,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

*55. *Given,* $a = b\cos C + c\cos B \ldots\ldots(1)$,

$b = c\cos A + a\cos C \ldots\ldots(2)$,

$c = a\cos B + b\cos A \ldots\ldots(3)$,

to deduce $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ *and the two similar formulae.*

a (1) + b (2) − c (3) gives

$$a^2 + b^2 - c^2 = (ab\cos C + ac\cos B) + (bc\cos A + ab\cos C) - (ac\cos B + bc\cos A)$$

$$= 2ab\cos C,$$

$$\therefore\ \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Similarly the values of $\cos B$ and $\cos A$ can be deduced.

*56. *Given,* $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, *to deduce* $a = b\cos C + c\cos B$, *&c.*

$$\sin A = \sin(180^\circ - A) = \sin(B+C) = \sin B \cos C + \cos B \sin C.$$

$$\therefore \frac{\sin A}{\sin B} = \cos C + \cos B . \frac{\sin C}{\sin B};$$

but, by hypothesis, $\frac{\sin A}{\sin B} = \frac{a}{b}$ and $\frac{\sin C}{\sin B} = \frac{c}{b}$,

$$\therefore \frac{a}{b} = \cos C + \cos B . \frac{c}{b},$$

$$\therefore a = b \cos C + c \cos B.$$

*57. *Given,* $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ *and the two similar expressions, to deduce* $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

$$\sin^2 A = 1 - \cos^2 A = 1 - \frac{(b^2 + c^2 - a^2)^2}{4b^2c^2}$$

$$= \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{4b^2c^2},$$

$$\therefore \frac{\sin A}{a} = \frac{+\sqrt{(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)}}{2abc}.$$

The sign + is placed before the radical sign because A is less than 180°, and therefore $\sin A$ is positive.

From the values of $\cos B$ and $\cos C$ we should obtain exactly the same values for $\frac{\sin B}{b}$ and $\frac{\sin C}{c}$ as the above value of $\frac{\sin A}{a}$; therefore $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

58. *To express the sine, cosine, and tangent of half an angle of a triangle in terms of the sides.*

Let $$2s = a + b + c,$$

$$\therefore 2(s-a) = b + c - a,$$

$$2(s-b) = c + a - b,$$

$$2(s-c) = a + b - c.$$

By Art. (54) we have

$$\cos A = \frac{b^2+c^2-a^2}{2bc},$$

$$\therefore\ 1-\cos A = 1-\frac{b^2+c^2-a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc} = \frac{(a-b+c)(a+b-c)}{2bc}$$
$$= \frac{2(s-b)(s-c)}{bc},$$

and
$$1+\cos A = 1+\frac{b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}$$
$$= \frac{2s(s-a)}{bc}.$$

Now $1-\cos A = 2\sin^2\frac{1}{2}A$, and $1+\cos A = 2\cos^2\frac{1}{2}A$, (Art. 37);

$$\therefore\ \sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

and
$$\cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}.$$

Also
$$\tan\tfrac{1}{2}A = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \sqrt{\frac{(s-b)(s-c)}{bc}} \div \sqrt{\frac{s(s-a)}{bc}},$$

$$\therefore\ \tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

The positive sign must be given to all the above radicals, since $\frac{1}{2}A$ is less than 90°, and therefore all its Trigonometrical Ratios are positive.

Cor. Hence $\sin A = 2\cos\frac{1}{2}A\sin\frac{1}{2}A$

$$= 2\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{(s-b)(s-c)}{bc}} = \frac{2}{bc}\cdot\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

59. *To investigate the three following formulae:*

$$\tan\tfrac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\tfrac{1}{2}A,$$

$$\sin\tfrac{1}{2}(B-C) = \frac{b-c}{a}\cos\tfrac{1}{2}A,$$

$$\cos\tfrac{1}{2}(B-C) = \frac{b+c}{a}\sin\tfrac{1}{2}A.$$

We have $\dfrac{\sin B}{\sin C}=\dfrac{b}{c}$,

$$\therefore\ \frac{\sin B-\sin C}{\sin B+\sin C}=\frac{b-c}{b+c},$$

$$\therefore\ \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}=\frac{b-c}{b+c}\ \text{(Art. 36)},$$

$$\therefore\ \tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\tfrac{1}{2}A,$$

since $\tan\frac{1}{2}(B+C)=\tan\frac{1}{2}(180^\circ-A)=\tan(90^\circ-\frac{1}{2}A)=\cot\frac{1}{2}A$.

Again, $\dfrac{b}{a}=\dfrac{\sin B}{\sin A}$ and $\dfrac{c}{a}=\dfrac{\sin C}{\sin A}$,

$$\therefore\ \frac{b-c}{a}=\frac{\sin B-\sin C}{\sin A},$$

and $\dfrac{b+c}{a}=\dfrac{\sin B+\sin C}{\sin A}$.

Now
$$\frac{\sin B-\sin C}{\sin A}=\frac{\sin B-\sin C}{\sin(B+C)}=\frac{2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B-C)}{2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B+C)}$$
$$=\frac{\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}A},$$

and
$$\frac{\sin B+\sin C}{\sin A}=\frac{\sin B+\sin C}{\sin(B+C)}=\frac{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B+C)}$$
$$=\frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A},$$

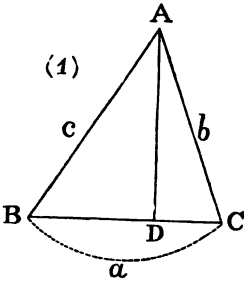
$$\therefore\ \frac{b-c}{a}=\frac{\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}A}\ \text{and}\ \frac{b+c}{a}=\frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}.$$

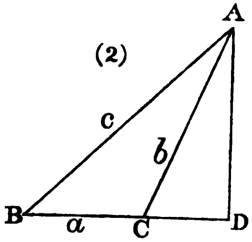
Therefore

$$\sin\tfrac{1}{2}(B-C)=\frac{b-c}{a}\cos\tfrac{1}{2}A\ \text{and}\ \cos\tfrac{1}{2}(B-C)=\frac{b+c}{a}\sin\tfrac{1}{2}A.$$

60. *To find the area of a triangle,* (1) *in terms of two sides and the included angle,* (2) *in terms of the three sides,* (3) *in terms of a side and the three angles.*

Put S for the area of the triangle ABC.





Since a triangle is half the rectangle under its base and altitude (Euclid I. 41), we have

$$S = \tfrac{1}{2}a \,.\, AD;$$

but $AD = b \sin C$,

$$\therefore\ S = \tfrac{1}{2}ab \sin C \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

By Art. (58), $\sin C = \dfrac{2}{ab}\sqrt{\{s(s-a)(s-b)(s-c)\}}$,

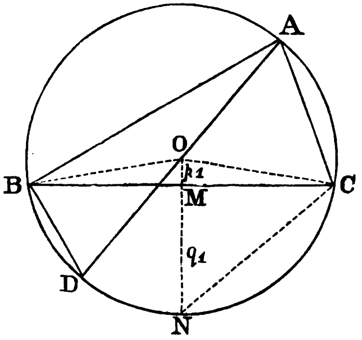
$$\therefore\ S = \sqrt{\{s(s-a)(s-b)(s-c)\}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{b}{a} = \frac{\sin B}{\sin A},\ \therefore\ b = a\frac{\sin B}{\sin A}.$$

Substituting this value of b in (1), we have

$$S = \tfrac{1}{2}a \sin C \,.\, a\frac{\sin B}{\sin A} = \frac{a^2 \sin B \sin C}{2 \sin A} \ldots\ldots\ldots\ldots(3).$$

61. *To find the radius* (R) *of the circle described about the triangle* ABC.



Draw the diameter AD, and join BD; then the angle ABD in a semicircle is a right angle, and the angle ABD equal to C.

From the right-angled triangle ADB we have

$$AB = AD \sin ADB,$$

that is, $c = 2R \sin C$;

$$\therefore\ R = \frac{c}{2 \sin C} = \frac{b}{2 \sin B} = \frac{a}{2 \sin A} \qquad \text{.........................(1)},$$

$$\sin A = \frac{2S}{bc} \text{ (Art. 58)},$$

$$\therefore\ R = \frac{a}{2 \sin A} = \frac{abc}{4S} \qquad \text{.....................................(2)}.$$

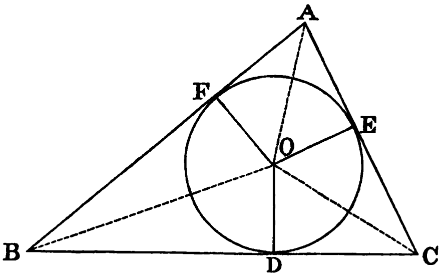
By (3) of the last Article,

$$S = \frac{a^2}{4 \sin^2 A} \cdot 2 \sin A \sin B \sin C,$$

$$\therefore\ S = 2R^2 \sin A \sin B \sin C \qquad \text{....................(3)},$$

by substituting from (1).

62. *To find the radius* (r) *of the circle inscribed in the triangle* ABC.



Let O be the centre of the inscribed circle, D, E, F its points of contact with the sides of the triangle; then

$$\text{triangle } BOC = \tfrac{1}{2} BC \cdot OD = \tfrac{1}{2} ar,$$
$$\text{triangle } COA = \tfrac{1}{2} CA \cdot OE = \tfrac{1}{2} br,$$
$$\text{triangle } AOB = \tfrac{1}{2} AB \cdot OF = \tfrac{1}{2} cr;$$

therefore, by addition,

$$\text{triangle } ABC = S = \tfrac{1}{2}(a + b + c)r = sr,$$

$$\therefore\ r = \frac{S}{s} \qquad \text{...........(1)}.$$

Again, $BD = DO \cot OBD = r \cot \frac{1}{2}B$ (See Euclid IV. 4),

$$DC = DO \cot OCD = r \cot \frac{1}{2}C;$$

$$\therefore\ BD + DC = a = r(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C)$$

$$= r \cdot \frac{\cos \frac{1}{2}B \sin \frac{1}{2}C + \sin \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}B \sin \frac{1}{2}C}$$

$$= r \cdot \frac{\sin \frac{1}{2}(B + C)}{\sin \frac{1}{2}B \sin \frac{1}{2}C}$$

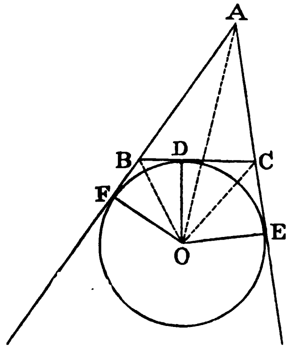
$$= r \cdot \frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}B \sin \frac{1}{2}C};$$

$$\therefore\ r = a \cdot \frac{\sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} \ \ldots\ldots\ldots\ldots\ldots (2).$$

63. *To find the radius* (r_1) *of the circle escribed to the side* a *of the triangle* ABC.

A circle is said to be escribed to a side of a triangle when it touches that side and the other two sides produced.

Let O be the centre of the escribed circle which touches



BC in D and the other two sides produced in E and F;

then quadrilateral $ABOC = \triangle\, AOB + \triangle\, AOC$

$$= \tfrac{1}{2}cr_1 + \tfrac{1}{2}br_1;$$

also quadrilateral $ABOC = \triangle\, ABC + \triangle\, BOC$

$$= S + \tfrac{1}{2}ar_1;$$

$$\therefore \quad \tfrac{1}{2}cr_1 + \tfrac{1}{2}br_1 = S + \tfrac{1}{2}ar_1,$$

$$\therefore \quad \tfrac{1}{2}r_1(c+b-a) = S,$$

$$\text{or} \quad r_1(s-a) = S,$$

$$\therefore \quad r_1 = \frac{S}{s-a}.$$

Similarly, $r_2 = \dfrac{S}{s-b}$ and $r_3 = \dfrac{s}{s-c}$,

where r_2, r_3 are the radii of the circles escribed to the sides b and c respectively.

64. In the figure of Art. (63) it is clear that the angle $OBD = 90° - \tfrac{1}{2}B$ and $OCD = 90° - \tfrac{1}{2}C$,

$\therefore\ BD = r_1 \cot(90° - \tfrac{1}{2}B) = r_1 \tan\tfrac{1}{2}B$,

and $CD = r_1 \cot(90° - \tfrac{1}{2}C) = r_1 \tan\tfrac{1}{2}C$;

$$\therefore\ BD + CD = a = r_1(\tan\tfrac{1}{2}B + \tan\tfrac{1}{2}C)$$

$$= r_1 \cdot \frac{\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C + \cos\tfrac{1}{2}B\sin\tfrac{1}{2}C}{\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C}$$

$$= r_1 \cdot \frac{\sin\tfrac{1}{2}(B+C)}{\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C}$$

$$= r_1 \cdot \frac{\cos\tfrac{1}{2}A}{\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C},$$

$$\therefore\ r_1 = a \cdot \frac{\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C}{\cos\tfrac{1}{2}A}.$$

Similarly, $r_2 = b \cdot \dfrac{\cos\tfrac{1}{2}C\cos\tfrac{1}{2}A}{\cos\tfrac{1}{2}B}$,

$$r_3 = c \cdot \frac{\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B}{\cos\tfrac{1}{2}C}.$$

65. *To prove* $r_1 + r_2 + r_3 - r = 4R$.

We have $r_1 + r_2 = \dfrac{S}{s-a} + \dfrac{S}{s-b} = \dfrac{S}{(s-a)(s-b)}(s-b+s-a)$

$$= \frac{Sc}{(s-a)(s-b)},$$

and $$r_3 - r = \frac{S}{s-c} - \frac{S}{s} = \frac{S}{s(s-c)}(s-s+c) = \frac{Sc}{s(s-c)},$$

$$\therefore\ r_1 + r_2 + r_3 - r = \frac{Sc}{s(s-a)(s-b)(s-c)}\{s(s-c)+(s-a)(s-b)\}$$

$$= \frac{c}{S}\{2s^2 - s(a+b+c) + ab\} = \frac{abc}{S} = 4R,$$

(Art. 61). Q.E.D.

*66. Let p_1, p_2, p_3, be the perpendiculars from the centre of the circumscribed circle of the triangle ABC on the sides a, b, c respectively, and q_1, q_2, q_3 the parts of these perpendiculars produced between the sides and circumscribed circle; then (fig. Art. 61) $q_1 = MC \tan MCN = \frac{1}{2}a \tan\frac{1}{2}A$, and

$r = AE \tan OAE = (s-a)\tan\frac{1}{2}A = (s-b)\tan\frac{1}{2}B = (s-c)\tan\frac{1}{2}C$, (see fig. Art. 62)

$r_1 = AE \tan OAE = s\tan\frac{1}{2}A$. (See fig. Art. 63.)

Similarly, $r_2 = s\tan\frac{1}{2}B$ and $r_3 = s\tan\frac{1}{2}C$;

$\therefore\ r_1 - r = s\tan\frac{1}{2}A - (s-a)\tan\frac{1}{2}A = a\tan\frac{1}{2}A$,

$\therefore\ 2q_1 = r_1 - r$, and $\therefore\ 2p_1 = 2R - 2q_1 = 2R + r - r_1$, with similar expressions for $2p_2$ and $2p_3$.

Similarly, $2q_2 = r_2 - r$, and $2q_3 = r_3 - r$;

$\therefore\ 2(q_1 + q_2 + q_3) = r_1 + r_2 + r_3 - 3r = 4R - 2r$, by Art. (65),

$\therefore\ q_1 + q_2 + q_3 = 2R - r$, and

$\therefore\ p_1 + p_2 + p_3 = 3R - (q_1 + q_2 + q_3) = R + r$.

*67. If D, D_1, D_2, D_3 be the distances of the centre of the circumscribed circle of the triangle ABC from the centres of the four circles touching the sides and sides produced, then

$$D^2 = R^2 - 2Rr,$$
$$D_1^2 = R^2 + 2Rr_1,$$
$$D_2^2 = R^2 + 2Rr_2,$$
$$D_3^2 = R^2 + 2Rr_3.$$

These formulae may be proved by help of the results

$$r = a \cdot \frac{\sin\frac{1}{2}B \sin\frac{1}{2}C}{\cos\frac{1}{2}A} \text{ and } r_1 = a \cdot \frac{\cos\frac{1}{2}B \cos\frac{1}{2}C}{\cos\frac{1}{2}A}$$

deduced in Arts. (62) and (64); but for simpler proofs, the reader is referred to "M'Dowell's Exercises on Euclid and in Modern Geometry," No. 97, page 80.

*68. *To prove that the distance* (δ_1) *between the centre of the inscribed circle of the triangle* ABC *and the centre of the circle escribed to the side* a *is* $a \sec\frac{1}{2}A$.

In the fig. of Art. 63, $AE = s$, and if the inscribed circle touch AC in E', then $AE' = s - a$, $\therefore E'E = s - (s - a) = a$.

Now $E'E = \delta_1 \cos OAE$,

that is, $a = \delta_1 \cos\frac{1}{2}A$, $\therefore \delta_1 = a \sec\frac{1}{2}A$.

Similarly, $\delta_2 = b \sec\frac{1}{2}B$,

and $\delta_3 = c \cdot \sec\frac{1}{2}C$.

*CHAPTER VII.

SOLUTION OF TRIANGLES.

69. The student is supposed to be already acquainted with "the nature and use of logarithms," from the Algebra.

In most books of Mathematical Tables are registered the values of the sines and cosines of all angles from 0° to 90° at intervals of 1′. These are called Natural Sines and Natural Cosines respectively. The logarithms of the Trigonometrical Ratios are increased by 10, in order to avoid *negative* characteristics.

These *increased* values are called the Tabular Logarithmic Sines, Cosines, Tangents, Cotangents, Secants, and Cosecants, and are registered in most Tables for all angles from 0° to 90° at intervals of 1′.

The Tabular Logarithms are usually denoted by L, and the ordinary ones by *log*; thus we have the relations

$$\log \sin A = L \sin A - 10, \ \log \tan A = L \tan A - 10, \text{ and so on.}$$

The mode of using the Tables is explained fully in every collection of Tables.

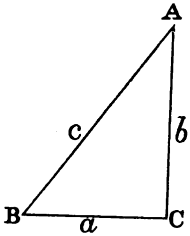
It should be noted that in the *first* quadrant, the sine, tangent, secant, Lsin, Ltan and Lsec *increase* as the angle *increases*, and the cosine, cotangent, cosecant, Lcos, Lcot, and Lcosec *decrease* as the angle *increases*.

70. When an angle of a triangle is determined from its

sine or *cosecant*, since there are *two* angles *each* less than 180°, which have the given sine or cosecant, it *may* be doubtful which of those angles is to be taken, or *both* may answer the conditions of the problem; but when an angle of a triangle is determined from its cosine, tangent, cotangent or secant, no ambiguity can exist respecting it, since there is only *one* angle less than 180°, which has a given cosine, tangent, cotangent or secant.

RIGHT-ANGLED TRIANGLES.

71. *Given the hypotenuse* (c) *and an acute angle* (A), *to solve the triangle.*



$\therefore\ B = 90° - A.$

$$a = c\sin A,\quad \therefore\ \log a = \log c + \log\sin A = \log c + L\sin A - 10,$$
$$b = c\cos A,\quad \therefore\ \log b = \log c + \log\cos A = \log c + L\cos A - 10.$$

These formulae determine B, a, and b.

72. *Given the hypotenuse* (c) *and a side* (a), *to solve the triangle.*

$$a = c\sin A,\quad \therefore\ \log a = \log c + L\sin A - 10,$$
$$\therefore\ L\sin A = 10 + \log a - \log c,$$

which determines A, and then $B = 90° - A$.

Now $b = c\cos A$ gives $\log b = \log c + L\cos A - 10$,

or, thus, $b^2 = c^2 - a^2 = (c - a)(c + a)$,

$$\therefore\ 2\log b = \log(c - a) + \log(c + a).$$

Either of these two formulae determines b.

In the above, though A is determined from its *sine*, there is no ambiguity, since the *acute* value must be taken in *a right-angled triangle.*

73. *Given a side* (a) *and an acute angle* (A), *to solve the triangle.*

$\therefore\ B = 90° - A,$

$$a = c\sin A, \quad \therefore\ \log a = \log c + L\sin A - 10,$$
$$\therefore\ \log c = 10 + \log a - L\sin A,$$
$$b = a\cot A, \quad \therefore\ \log b = \log a + L\cot A - 10.$$

The above formulae determine B, c, b.

74. *Given the two sides* (a and b), *to solve the triangle.*

$$\tan A = \frac{a}{b}, \quad \therefore\ L\tan A - 10 = \log a - \log b,$$
$$\therefore\ L\tan A = 10 + \log a - \log b, \text{ and } \therefore\ B = 90° - A.$$
$$a = c\sin A, \quad \therefore\ \log a = \log c + L\sin A - 10,$$
$$\therefore\ \log c = 10 + \log a - L\sin A;$$

or $c = \sqrt{(a^2 + b^2)}$, but this value of c cannot be *directly* computed by logarithms.

The above formulae determine A, B, c.

OBLIQUE-ANGLED TRIANGLES.

75. Case I. *Given two angles* (A and B) *and a side* (a), *to solve the triangle.*

$\therefore\ C = 180° - (A + B),$

$$b = a\frac{\sin B}{\sin A}, \quad \therefore\ \log b = \log a + L\sin B - L\sin A,$$
$$c = a\frac{\sin C}{\sin A}, \quad \therefore\ \log c = \log a + L\sin C - L\sin A.$$

The above formulae determine C, b, c.

76. Case ii. *Given two sides* (b and c) *and the included angle* (A), *to solve the triangle.*

Here we have (Art. 59.)

$$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\tfrac{1}{2}A,$$

$$\sin\tfrac{1}{2}(B-C)=\frac{b-c}{a}\cos\tfrac{1}{2}A,$$

$$\cos\tfrac{1}{2}(B-C)=\frac{b+c}{a}\sin\tfrac{1}{2}A;$$

$$\therefore\ L\tan\tfrac{1}{2}(B-C)=L\cot\tfrac{1}{2}A+\log(b-c)-\log(b+c),$$

$$\log a=L\cos\tfrac{1}{2}A+\log(b-c)-L\sin\tfrac{1}{2}(B-C),$$

$$\log a=L\sin\tfrac{1}{2}A+\log(b+c)-L\cos\tfrac{1}{2}(B-C).$$

The first formula will determine $\frac{1}{2}(B-C)$; then, since $\frac{1}{2}(B+C)=90^\circ-\frac{1}{2}A$ is known, B and C will become known.

The second or third formula will now determine a;

or a may be found thus,

$$\frac{a}{b}=\frac{\sin A}{\sin B},\quad \therefore\ \log a=\log b+L\sin A-L\sin B;$$

but it is generally preferable to find a from either of the other two formulae, as only *two* new logarithms are required, whereas this last method requires *three* new logarithms.

77. Case iii. *Given two sides* (a and b), *and an angle* (A) *opposite to one of them, to solve the triangle.*

$$\sin B=\frac{b}{a}\sin A,$$

$$\therefore\ L\sin B=L\sin A+\log b-\log a.$$

This formula determines *two supplemental values* for B.

When the given angle A is right or obtuse, the *acute* value of B must be taken; also when b is less than a, then B is less than A, and therefore B must be *acute.* When $a=b\sin A$, then $\sin B=1$, and therefore $B=90^\circ$.

Therefore in all these cases there is only *one* solution.

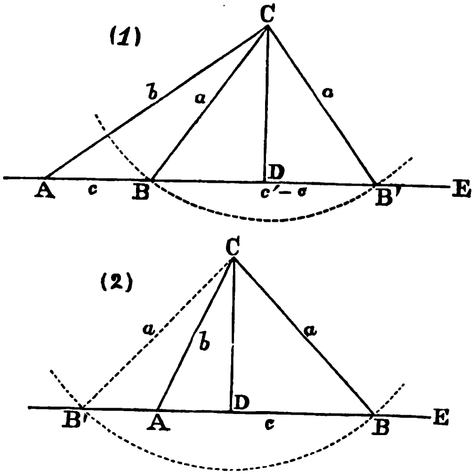
When a is less than $b\sin A$, then $\sin B$ is greater than 1, and a solution is therefore impossible.

When A is *acute* and a less than b, then B is greater than A, and B may therefore be obtuse; hence there are *two* solutions in this case, *provided that* $b\sin A$ be *less than* a.

The solution may now be completed by Case I.

The above results may also be deduced from the Geometrical Construction, thus,

Draw $AC=b$, the indefinite straight line AE making the angle $CAE=A$, and CD perpendicular to AE.



From the centre C at a distance $=a$ describe a circle which will *generally* cut AE in the two points B and B'.

When B, B' are on the same side of A as in fig. (1), it is clear that there are *two* triangles with the given data, viz. ABC and $AB'C$, having their angles ABC and $AB'C$ *supplemental*.

In this case a is obviously less than b and A *acute*.

When B and B' are on different sides of A, as in fig. (2), there is only *one* triangle ABC with the given data. In this case a is clearly greater than b, and therefore A may be either acute, right, or obtuse, but B must be *acute*.

Therefore when the side opposite the given angle is less than the other given side, there are *generally two* solutions; and when it is greater than the other side, there is only *one* solution.

When the side opposite the given angle $= CD$, that is, when $a = b \sin A$, then the two points B and B' coincide with D, the circle touches AE in D, and there is only *one* solution.

If the circle do not meet AE, or a be less than CD, there is *no* solution.

CASE III. is generally called the *Ambiguous Case*.

78. When there are *two* solutions in Case III., if a, b, c and A, B, C be the sides and angles of one triangle, a, b, c' and A, B', C' the corresponding sides and angles of the other, then we may find c', B', C' in terms of a, b, c, A, B, C, thus,

$$B' = 180° - B,$$

and $$C' - C = B - B' = B - (180° - B) = 2B - 180°,$$

$$\therefore \quad C' = 2B + C - 180°.$$

Also $c = b \cos A + a \cos B$,

$$c' = b \cos A + a \cos B' = b \cos A - a \cos B;$$

$$\therefore \quad cc' = b^2 \cos^2 A - a^2 \cos^2 B = b^2 - a^2 - b^2 \sin^2 A + a^2 \sin^2 B = b^2 - a^2,$$

$$\therefore \quad c' = \frac{b^2 - a^2}{c} = \frac{(b-a)(b+a)}{c};$$

or $$\frac{c'}{a} = \frac{\sin C'}{\sin A} = \frac{\sin(2B + C - 180°)}{\sin A},$$

$$\therefore \quad c' = \frac{a \sin(2B + C - 180°)}{\sin A}.$$

79. Case IV. *Given the three sides, to solve the triangle.*

Here we have

$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\therefore \quad L\sin\tfrac{1}{2}A = 10 + \tfrac{1}{2}\{\log(s-b) + \log(s-c) - \log b - \log c\};$$

$$\cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}},$$

$$\therefore \quad L\cos\tfrac{1}{2}A = 10 + \tfrac{1}{2}\{\log s + \log(s-a) - \log b - \log c\};$$

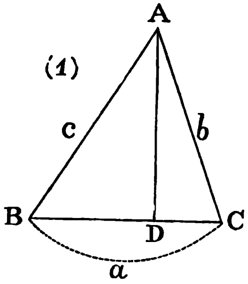
$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\therefore \quad L\tan\tfrac{1}{2}A = 10 + \tfrac{1}{2}\{\log(s-b) + \log(s-c) - \log s - \log(s-a)\},$$

with similar formulae for the other half angles.

We may also solve this case thus:

Suppose BC or a the *greater* side, and draw AD perpen-



dicular to BC; thus the angles B and C are both *acute*, and $BD^2 - CD^2 = c^2 - b^2$, or $(BD - CD)(BD + CD) = (c-b)(c+b)$,

$$\therefore \quad BD - CD = \frac{(c-b)(c+b)}{a} = k, \text{ suppose},$$

$$\therefore \quad BD = \frac{a+k}{2} \text{ and } CD = \frac{a-k}{2} \text{ are known.}$$

We have now the hypotenuse and a side in each of the two right-angled triangles ABD and ACD, and we can therefore find the angles by Art. (72).

I will now add the solutions of a few Examples to illustrate the use of the preceding formulae.

80. Case I. Given $A = 57° 17' 20''$, $B = 102° 13'$, and $a = 97$ yards; find C, b, and c.

Here

$$C = 180° - (A + B) = 180° - (57° 17' 20'' + 102° 13') = 20° 29' 40'',$$

$$\log b = \log a + L \sin B - L \sin A,$$

$$\log c = \log a + L \sin C - L \sin A.$$

To find $L \sin A$ we take from the Tables

$$\left.\begin{array}{l} L \sin 57° 18' = 9{\cdot}9250597 \\ L \sin 57° 17' = 9{\cdot}9249786 \end{array}\right\} \text{subtract,}$$

$$\cdot 0000811 = \text{difference for } 1' \text{ or } 60'';$$

but as the difference 812 stands in the Tables opposite $L \sin 57° 17'$ in the column marked D, it was unnecessary to take out $L \sin 57° 18'$, and this superfluous work will, in future, be avoided.

Now A exceeds $57° 17'$ by $20''$.

Therefore as $60'' : 20'' :: \cdot 0000811$: required difference for $20''$, $= \cdot 0000270$ nearly.

We must *add* this difference to $L \sin 57° 17'$, since the L sine increases as the angle increases;

$$\begin{array}{r} 9{\cdot}9249786 \\ 270 \\ \hline \end{array}$$

$$\therefore \quad L \sin A = L \sin 57° 17' 20'' = 9{\cdot}9250056.$$

To find $L \sin C$,

$$L \sin 20° 29' = 9{\cdot}5439873, \text{ difference for } 60'' = 3380$$

$$2253 = \text{difference for } 40'',$$

$$\therefore L \sin C = L \sin 20° 29' 40'' = 9{\cdot}5442126.$$

Also

$$L \sin B = L \sin 102° 13' = L \sin(180° - 102° 13') = L \sin 77° 47'.$$

Hence we may arrange the solution as follows:

$$\left.\begin{aligned} \log a &= \log 97 = 1.9867717 \\ L\sin B &= L\sin 77^\circ\, 47' = 9{\cdot}9900521 \end{aligned}\right\} \text{add,}$$

$$\left.\begin{aligned} & \quad 11{\cdot}9768238 \\ L\sin A &= L\sin 57^\circ\, 17'\, 20'' = 9{\cdot}9250056 \end{aligned}\right\} \text{subtract,}$$

$$\therefore \quad \log b = \log 112{\cdot}67257 = 2{\cdot}0518182.$$

$$\left.\begin{aligned} \log a &= \log 97 = 1{\cdot}9867717 \\ L\sin C &= L\sin 20^\circ\, 29'\, 40'' = 9{\cdot}5442126 \end{aligned}\right\} \text{add,}$$

$$\left.\begin{aligned} & \quad 11{\cdot}5309843 \\ L\sin A &= L\sin 57^\circ\, 17'\, 20'' = 9{\cdot}9250056 \end{aligned}\right\} \text{subtract,}$$

$$\therefore \quad \log c = \log 40{\cdot}36256 = 1{\cdot}6059787.$$

In finding b we proceed thus.

The nearest mantissa in the Tables to $\log b$ is ·0518083, opposite to which we find the number 11267 and the difference 385.

No.	Mantissae	D.
	·0518182	
11267	·0518083	385
	99.	

Therefore as 385 : 99 :: 1 : ·257, the figures to be placed after 11267,

therefore $b = 112{\cdot}67257$.

b must have *three* figures before the decimal point because the characteristic of its logarithm is 2.

The process for finding c is exactly the same as that for finding b.

We might have found $L\sin A - \log a$, and then subtracted the result from $L\sin B$ and $L\sin C$ successively, to find $\log b$ and $\log c$. This would have been more concise than the above process.

Ans.—$C = 20^\circ\, 29'\, 40''$, $b = 112{\cdot}67257$, and $c = 40{\cdot}36256$.

81. CASE II. *Given $b=73$, $c=55$, and $A=84°\,16'\,24''$; find B, C, and a.*

We have

$$L\tan\tfrac{1}{2}(B-C)=L\cot\tfrac{1}{2}A+\log(b-c)-\log(b+c)$$
$$=L\cot 42°\,8'\,12''+\log 18-\log 128;$$

$L\cot 42°\,8'=10{\cdot}0435306$, $2539=$ difference for $60''$,

$\therefore\quad \frac{12}{60}\times 2539=508=$ difference for $12''$.

508 must be subtracted from $L\cot 42°\,8'$, since $L\cot$ *decreases* as the angle *increases.*

$$\left.\begin{array}{r}10{\cdot}0435306\\ 508\end{array}\right\}\text{ subtract,}$$

$$\therefore\quad \left.\begin{array}{r}L\cot 42°\,8'\,12''=10{\cdot}0434798\\ \log 18=\ 1{\cdot}2552725\end{array}\right\}\text{ add,}$$

$$\left.\begin{array}{r}11{\cdot}2987523\\ \log 128=\ 2{\cdot}1072100\end{array}\right\}\text{ subtract,}$$

$$\therefore\quad L\tan\tfrac{1}{2}(B-C)=\ 9{\cdot}1915423$$
$$L\tan 8°\,50'=\ 9{\cdot}1914621,\quad 8318=\text{difference for }60''.$$

As $8318 : 802 :: 60'' : 6''$ nearly,

therefore $\tfrac{1}{2}(B-C)=8°\,50'\,6''$,

and $\tfrac{1}{2}(B+C)=90°-\tfrac{1}{2}A=47°\,51'\,48''$;

therefore $B=56°\,41'\,54''$,

and $C=39°\,1'\,42''$.

Also $\log a=\log b+L\sin A-L\sin B$

$$=\log 73+\sin 84°\,16'\,24''-L\sin 56°\,41'\,54'',$$

$$\left.\begin{array}{r}\log 73=1{\cdot}8633229\\ L\sin 84°\,16'\,24''=9{\cdot}9094554\end{array}\right\}\text{ add,}$$

$$\left.\begin{array}{r}11{\cdot}7727783\\ L\sin 56°\,41'\,54''=9{\cdot}9220980\end{array}\right\}\text{ subtract,}$$

$\therefore\quad \log a=1{\cdot}8506803.$

$\therefore\ a=70{\cdot}90556.$

If a had been found from either the second or third formula in Art. (76), only *two* new logarithms would have been necessary: we have used *three* in the above.

Ans.—$B=56°\,41'\,54''$, $C=39°\,1'\,42''$, and $a=70{\cdot}90556$.

82. CASE III. Given $a=97$, $b=100$, and $A=57°\,17'\,20''$; find B, C, and c.

Here a is greater than $b\sin A$ (but this will appear as we proceed with the solution from $L\sin B$ being *less than* 10), but less than b, and A is *acute*, therefore there are *two* triangles with the given data.

$$L\sin B = L\sin A + \log b - \log a$$
$$= L\sin 57°\,17'\,20'' + \log 100 - \log 97;$$

$$\left.\begin{array}{r} L\sin 57°\,17'\,20'' = 9{\cdot}9250056 \\ \log 100 = 2{\cdot} \end{array}\right\} \text{add,}$$

$$\left.\begin{array}{r} 11{\cdot}9250056 \\ \log 97 = 1{\cdot}9867717 \end{array}\right\} \text{subtract,}$$

$$\therefore \quad L\sin B = 9{\cdot}9382339;$$

$$\therefore \quad B = 60°\,9'\,40'', \text{ or } 180° - 60°\,9'\,40'' = 119°\,50'\,20'',$$

$$\therefore \quad C = 180° - (57°\,17'\,20'' + 60°\,9'\,40'') = 62°\,33',$$

$$\text{or} \quad C = 180° - (57°\,17'\,20'' + 119°\,50'\,20'') = 2°\,52'\,20''.$$

In the two triangles therefore we now know,

$$(1)\ \left\{\begin{array}{l} a=97, \\ b=100, \\ A=57°\,17'\,20'', \\ B=60°\,9'\,40'', \\ C=62°\,33'. \end{array}\right. \qquad (2)\ \left\{\begin{array}{l} a=97, \\ b=100, \\ A=57°\,17'\,20'', \\ B=119°\,50'\,20'', \\ C=2°\,52'\,20''. \end{array}\right.$$

In both cases the remaining part c may be found from the formula

$$\log c = \log a + L\sin C - L\sin A. \quad \text{(See Case I.)}$$

83. Case iv. *Given $a=66$, $b=40$, and $c=32$; find the angles.*

Here $2s=66+40+32=138$,

therefore $s=69$,

$s-a=3$,

$s-b=29$,

$s-c=37$.

$$L\tan\tfrac{1}{2}A=10+\tfrac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}$$
$$=\tfrac{1}{2}(20+\log 29+\log 37-\log 69-\log 3),$$
$$L\tan\tfrac{1}{2}B=10+\tfrac{1}{2}\{\log(s-a)+\log(s-c)-\log s-\log(s-b)\}$$
$$=\tfrac{1}{2}(20+\log 3+\log 37-\log 69-\log 29),$$
$$L\tan\tfrac{1}{2}C=\tfrac{1}{2}(20+\log 3+\log 29-\log 69-\log 37)$$

$$\left.\begin{aligned}20+\log 29&=21{\cdot}4623980\\ \log 37&=1{\cdot}5682017\end{aligned}\right\}\text{add,}\qquad \left.\begin{aligned}\log 69&=1{\cdot}8388491\\ \log 3&=0{\cdot}4771213\end{aligned}\right\}\text{add,}$$

$$\left.\begin{aligned}23{\cdot}0305997\\ 2{\cdot}3159704\end{aligned}\right\}\text{subtract,}\qquad 2{\cdot}3159704$$

$$2)\underline{20{\cdot}7146293}$$

$\therefore\quad L\tan\tfrac{1}{2}A=10{\cdot}3573146$

$L\tan 66^\circ 17'=10{\cdot}3572227$, $3430=$ difference for $60''$.

As $3430 : 919 :: 60'' : 16''$,

$\therefore\ \tfrac{1}{2}A=66^\circ 17' 16''$, and $A=132^\circ 34' 32''$.

$$\left.\begin{aligned}20+\log 3&=20{\cdot}4771213\\ \log 37&=1{\cdot}5682017\end{aligned}\right\}\text{add,}\qquad \left.\begin{aligned}\log 69&=1{\cdot}8388491\\ \log 29&=1{\cdot}4623980\end{aligned}\right\}\text{add,}$$

$$\left.\begin{aligned}22{\cdot}0453230\\ 3{\cdot}3012471\end{aligned}\right\}\text{subtract,}\qquad 3{\cdot}3012471$$

$$2)\underline{18{\cdot}7440759}$$

$\therefore\quad L\tan\tfrac{1}{2}B=9{\cdot}3720379$

$L\tan 13^\circ 15'=9{\cdot}3719333$, $5659=$ difference for $60''$.

As $5659 : 1046 :: 60'' : 11''$,

therefore $\tfrac{1}{2}B=13^\circ 15' 11''$,

therefore $B=26^\circ 30' 22''$.

$$\left.\begin{aligned} 20+\log 3 &= 20\cdot4771213 \\ \log 29 &= 1\cdot4623980 \end{aligned}\right\} \text{add,} \qquad \left.\begin{aligned} \log 69 &= 1\cdot8388491 \\ \log 37 &= 1\cdot5682017 \end{aligned}\right\} \text{add,}$$

$$\left.\begin{aligned} 21\cdot9395193 \\ 3\cdot4070508 \end{aligned}\right\} \text{subtract,} \qquad 3\cdot4070508$$

$$2)\underline{18\cdot5324685}$$

$$\therefore \quad L\tan\tfrac{1}{2}C = 9\cdot2662342$$

$$L\tan 10^\circ 27' = 9\cdot2658470, \quad 7077 = \text{difference for } 60''.$$

As 7077 : 3872 :: 60'' : 33'',

therefore $\tfrac{1}{2}C = 10^\circ\, 27'\, 33''$,

therefore $C = 20^\circ\, 55'\, 6''$.

Now $A+B+C = 132^\circ\, 34'\, 32'' + 26^\circ\, 30'\, 22'' + 20^\circ\, 55'\, 6'' = 180^\circ$.

We have calculated *all* the angles for the purpose of verifying the work by taking the sum of the three angles. This sum ought to be *very nearly* $= 180^\circ$ when the work is correct and the Tables accurate. We say *very nearly*, because we have only computed the angles to the nearest second.

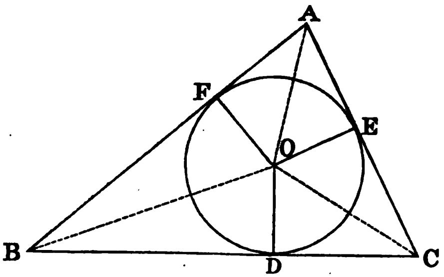
Of course when the correct values of two angles have been obtained, the other angle can be found by taking the sum of the two from 180°.

NOTE I. I will add here one or two propositions which are often useful in the solution of problems.

(1) If θ be the circular measure of a positive angle less than 90°, prove that $\sin\theta$, θ and $\tan\theta$ are in ascending order of magnitude, and hence shew that when θ is indefinitely diminished $\dfrac{\sin\theta}{\theta} = 1$, $\dfrac{\tan\theta}{\theta} = 1$.

Let AO meet the circumference in P, join PE and draw PM perpendicular to OE, and let θ equal circular measure of the angle POE; then $\dfrac{\text{arc } PE}{r} = \theta$, $\therefore$ arc $PE = r\theta$.

Also $\Delta\ OPE$, sector POE, and $\Delta\ AOE$ are in ascending



order of magnitude, that is, $\frac{1}{2}OE.PM$, $\frac{1}{2}OE.\text{arc}\,PE$, and $\frac{1}{2}OE.AE$, (see Art. 4),

or $\frac{1}{2}r^2\sin\theta$, $\frac{1}{2}r^2\theta$, and $\frac{1}{2}r^2\tan\theta$ are in ascending order of magnitude,

therefore $\qquad \sin\theta,\ \theta$ and $\tan\theta$,

and also 1, $\frac{\theta}{\sin\theta}$ and $\frac{1}{\cos\theta}$ are in ascending order of magnitude; but when θ is diminished indefinitely and ultimately vanishes, $\cos\theta = 1$,

$$\therefore\ \frac{\theta}{\sin\theta} \text{ ultimately} = 1, \text{ when } \theta = 0, \text{ and } \therefore\ \frac{\sin\theta}{\theta} = 1,$$

$$\therefore\ \frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta} = 1, \text{ when } \theta = 0.$$

(2) $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$, which is $< 1 - 2(\frac{1}{2}\theta)^2$, when $\theta < \frac{\pi}{2}$,

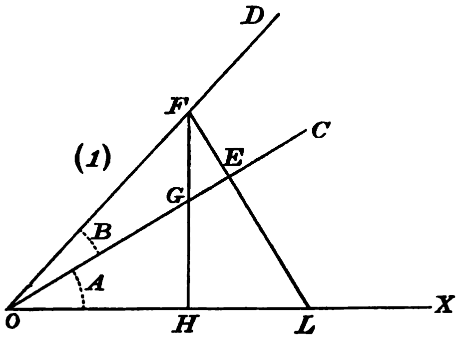
$\therefore\ \cos\theta < 1 - \frac{1}{2}\theta^2$, when θ is $< \frac{\pi}{2}$.

NOTE II.—The following neat proofs of the formulae

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

were communicated to me by W. H. Besant, Esq., M.A., formerly Fellow of St. John's College, too late for insertion in the text.

Draw LEF perpendicular to OC and FH to OX; then



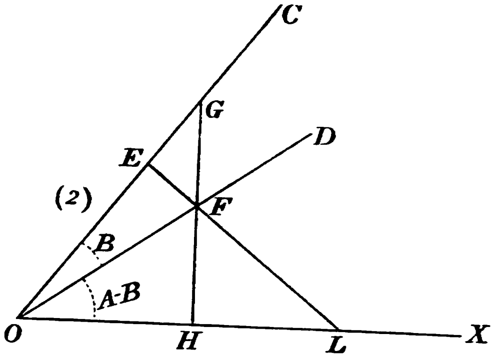
from the similar triangles FHL, GOH,

$$FH : FL :: OH : OG, \quad \therefore \frac{FH}{OH} = \frac{FL}{OG}.$$

Therefore $$\tan(A+B) = \frac{FH}{OH} = \frac{FL}{OG} = \frac{LE+EF}{OE-EG}$$

$$= \frac{\frac{LE}{OE} + \frac{EF}{OE}}{1 - \frac{EG}{FE} \cdot \frac{FE}{OE}}.$$

$$= \frac{\tan A - \tan B}{1 - \tan A \tan B}.$$



And $\tan(A-B)=\frac{FH}{OH}=\frac{FL}{OG}=\frac{LE-EF}{OE+EG}$

$$=\frac{\frac{LE}{OE}-\frac{EF}{OE}}{1+\frac{EG}{FE}\cdot\frac{FE}{OE}}$$

$$=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

EXAMPLES ON CHAPTER I., Page 1.

1. Reduce $29° 12' 18''$ to grades.

2. Reduce $37^g 14' 39''$ to degrees.

3. Reduce one sexagesimal minute to centesimal seconds.

4. Express the angles in Examples 1 and 2 in circular measure.

5. Find the number of degrees in an angle whose circular measure is $\frac{5}{4}$.

6. The radius of a circle is 14 feet; find its circumference and area, having given $\pi = \frac{22}{7}$.

7. The radius of a circle is 25 feet, and the angle of a sector of it has $\frac{3}{4}$ for its circular measure; find the area of the sector.

8. The numbers of the sides of two regular polygons are as 1 : 2, and the number of grades in all the interior angles of the one is to the number of degrees in all the interior angles of the other as 5 : 12; find the number of the sides of each polygon.

9. What is the Trigonometrical notion of an angle? Explain the different systems that are usually adopted for the measurement of angles, and compare the values of the units of measurement.

Shew that there are eleven and only eleven pairs of regular polygons, which are such that the number of degrees in an angle of one of them is equal to the number of grades in an angle of the other, and that there are only four pairs where these angles are expressed as integers.

EXAMPLES ON CHAPTER II., Page 7.

1. Given $\sin\alpha = \frac{4}{5}$, $\cos\beta = \frac{12}{13}$, $\tan\gamma = -2$, and $\sec\delta = 2.6$; find all the other Trigonometrical Ratios of these four angles.

2. Given $\sin\theta = a$; find all the other Trigonometrical Ratios of θ.

3. Given $\cos\theta = a$, $\tan\phi = \frac{a}{b}$, and $\sec\psi = a$; find all the other Trigonometrical Ratios of θ, ϕ, and ψ.

4. Express the Trigonometrical Ratios of the following angles by means of the Trigonometrical Ratios of positive angles less than $90°$:

$$330°, \quad 590°, \quad -370°, \quad \text{and } 1866°.$$

5. Obtain solutions of the following equations:

(1) $\cos^2\theta = \frac{3}{2}\sin\theta$, (3) $\cos^2\theta - 2\sin\theta + \frac{1}{4} = 0$,
(2) $\tan\theta = 2\sin\theta$, (4) $5\sec^4\theta - 8\sec^2\theta - 48 = 0$.

6. Trace the changes in $\sin\theta - \cos\theta$ and in $\sin^2\theta - \cos^2\theta$ as θ varies from 0 to 2π.

7. Discuss the equation $\sec^2\theta = \frac{4ab}{(a+b)^2}$.

EXAMPLES ON CHAPTER III., Page 20.

1. Given $\cos\theta = \frac{3}{5}$ and $\tan\phi = \frac{4}{3}$; construct the angles θ and ϕ geometrically.

2. Write down the general values of θ in the following equations:

(1) $\tan\theta = 1$, (4) $\sin\theta = -\frac{1}{2}$,
(2) $\sin\theta = 1$, (5) $\cos^2\theta = \sin^2\alpha$,
(3) $\cos\theta = 1$,

having given $\tan\frac{\pi}{4} = 1$, $\sin\frac{\pi}{2} = 1$, $\cos 0 = 1$, and $\sin\frac{\pi}{6} = \frac{1}{2}$.

3. Investigate a general formula for all angles the sines of which are equal to the sine of α.

Find the general value of an angle such that its cosine is to its tangent as 3 to 2.

EXAMPLES ON CHAPTERS IV. AND V., Pp. 24, 35.

Prove the three following equations:

1. $\dfrac{\cos A - \cos 3A}{\sin A + \sin 3A} = \tan A.$

2. $\dfrac{\cos A - 2\cos 3A + \cos 5A}{\sin A + 2\sin 3A + \sin 5A} = -\tan^2 A \cot 3A.$

3. $\tan(45^\circ + A) - (\tan 45^\circ - A) = 2\tan 2A.$

4. Reduce to its simplest value

$$\cos\theta + \cos\left(\frac{2\pi}{3} - \theta\right) + \cos\left(\frac{4\pi}{3} - \theta\right).$$

5. Reduce to its simplest form the expression

$$2\cos^2\theta + \cos^2 2\theta - 2\cos^2\theta\cos 2\theta.$$

6. Simplify the expression

$$\cos^2\theta + \cos^2\phi + \cos^2(\theta + \phi) - 2\cos\theta\cos\phi\cos(\theta + \phi).$$

7. Prove that

$$\tfrac{1}{2}\tan\frac{\theta}{2} + \tfrac{1}{4}\tan\frac{\theta}{4} = \tfrac{1}{4}\cot\frac{\theta}{4} - \cot\theta.$$

Hence prove that, θ being the circular measure of an angle,

$$\cot\theta + \tfrac{1}{2}\tan\frac{\theta}{2} + \tfrac{1}{4}\tan\frac{\theta}{4} + \ldots\ldots = \frac{1}{\theta}.$$ (See Note I., page 69.)

8. Given $\sin\theta + \cos\theta = \dfrac{1}{\sqrt{2}}$; find θ.

9. Given $\tan\left(\dfrac{\pi}{4} - \theta\right) + \tan\left(\dfrac{\pi}{4} + \theta\right) = 4$; find θ.

10. Given $\sin 210^\circ = -\tfrac{1}{2}$; find $\sin 105^\circ$ and $\cos 105^\circ$.

11. Given $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$; find $\tan 165^\circ$.

12. Given $\tan\frac{1}{2}A = 2 - \sqrt{3}$; find $\sin A$.

13. Solve the equations:

(1) $(1+\sin\theta)(1-2\sin\theta)^2 = (1-\cos\alpha)(1+2\cos\alpha)^2$.

(2) $\dfrac{\sin\alpha\cos(\beta+\theta)}{\sin\beta\cos(\alpha+\theta)} = \dfrac{\tan\beta}{\tan\alpha}$.

(3) If $\tan^2 A\tan A' = \tan^2 B\tan B' = \tan^2 C\tan C'$
$= \tan A\tan B\tan C$.

and $\operatorname{cosec} 2A + \operatorname{cosec} 2B + \operatorname{cosec} 2C = 0$,

then will $\tan(A-A') = \tan(B-B') = \tan(C-C')$.

14. Prove by a Geometrical construction, that

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B};$$

and deduce the expansion of $\cos(A-B)$ from this formula.

15. If $\alpha+\beta+\gamma = \dfrac{\pi}{2}$, prove that

$$\cos\alpha+\cos\beta+\cos\gamma = 4\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\gamma}{2}\right),$$

and

$$\sin\alpha+\sin\beta+\sin\gamma-1 = 4\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\gamma}{2}\right).$$

16. Prove the following formulae:

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B};$$

$$\cos A - \cos B = 2\sin\frac{B-A}{2}\sin\frac{B+A}{2};$$

$$\text{and } \frac{1-\tan\frac{1}{2}\alpha}{1+\tan\frac{1}{2}\alpha} = \tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).$$

If $\alpha+\beta+\gamma = \dfrac{\pi}{2}$, prove that

$$\frac{\left(1-\tan\frac{\alpha}{2}\right)\left(1-\tan\frac{\beta}{2}\right)\left(1-\tan\frac{\gamma}{2}\right)}{\left(1+\tan\frac{\alpha}{2}\right)\left(1+\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\gamma}{2}\right)} = \frac{\sin\alpha+\sin\beta+\sin\gamma-1}{\cos\alpha+\cos\beta+\cos\gamma}.$$

If an angle be divided into two equal parts and also into two unequal parts, the rectangle of the sines of the unequal parts, together with the square of the sine of the angle between the dividing lines, is equal to the square of the sine of half the angle.

17. Investigate the different values of $\cos\frac{n\pi}{5}$ for positive integral values of n.

Shew that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\left(\frac{1}{2}\right)^7.$$

18. Find the value of θ so that

$$\sec\theta,\ \sec\left(\frac{2\pi}{3}+\theta\right),\quad \sec\left(\frac{2\pi}{3}-\theta\right)$$

may be in Arithmetical progression, and calculate the common difference.

Examples on Chapter VI., Page 44.

1. In any triangle ABC, prove that

$$\sin A+\sin B+\sin C=4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C,$$

and $$\sin B+\sin C-\sin A=4\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

2. Prove $S=\dfrac{2abc}{a+b+c}\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$

3. Prove $$2R+2r=a\cot A+b\cot B+c\cot C,$$

and $$a\cos A+b\cos b+c\cos C=4R\sin A\sin B\sin C.$$

4. Prove $$r=4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C,$$

and $$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}.$$

5. Perpendiculars are drawn from the angles A, B, C of a triangle to the opposite sides and produced to meet the

circumscribed circle; if α, β, γ be these produced parts, prove that

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2\tan A\tan B\tan C.$$

6. If r be the radius of the circle inscribed in the triangle ABC, and r_a, r_b, r_c the radii of the circles inscribed between this circle and the sides containing the angles A, B, C respectively; prove that

$$r_a=r\,.\,\frac{1-\sin\frac{1}{2}A}{1+\sin\frac{1}{2}A}=r\,.\,\tan^2\frac{\pi-A}{4}=r\cot^2\frac{\pi+A}{4},$$

and hence shew that $\sqrt{(r_a r_b)}+\sqrt{(r_b r_c)}+\sqrt{(r_c r_a)}=r.$

7. If A be area of the inscribed circle of a triangle, A_1, A_2, A_3 the areas of the three escribed circles, prove that

$$\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{A}}.$$

8. Prove that $\sin A+\sin B+\sin C=\dfrac{s}{R}$,

and $$\sin B+\sin C-\sin A=\frac{s-a}{R}.$$

9. If p be the perpendicular from A on a, prove that

$$p=\frac{a}{\cot B+\cot C}=\frac{a\sin B\sin C}{\sin A}.$$

10. Prove $R=\dfrac{a\cos A+b\cos B+c\cos C}{\sin 2A+\sin 2B+\sin 2C}.$

11. In any triangle ABC, prove that

$$a^2+b^2-2ab\cos(60^\circ+C)=b^2+c^2-2bc\cos(60^\circ+A)$$
$$=c^2+a^2-2ca\cos(60^\circ+B).$$

12. Investigate an expression for the cosine of the angle of a triangle in terms of its sides.

The sides of a given triangle are a, b, c, and the angles

α, β, γ; prove that, if a point be taken within an equilateral triangle, whose distances from its angles are proportional to a, b, c, the angles between these distances will be

$$\frac{\pi}{3}+\alpha,\ \frac{\pi}{3}+\beta,\ \frac{\pi}{3}+\gamma, \text{ respectively.}$$

13. Investigate expressions for the radii of the circles that touch the sides of a given triangle.

If the circle which touches the sides AB, AC produced touch BC at P, prove that, with the usual notation,

$$a(s^2 - AP^2) = 4s(s-b)(s-c).$$

14. If the sides of a triangle be in arithmetical progression, the cotangents of its semiangles shall also be in arithmetical progression.

15. If the points of contact of the four circles touching the sides of a triangle ABC be joined, and the area of the four triangles thus formed be denoted by Δ, Δ_1, Δ_2, Δ_3; prove that

$$\Delta = \tfrac{1}{2}r^2(\sin A + \sin B + \sin C) = \frac{rS}{2R} = 2r^2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C,$$

$$\Delta_1 = \tfrac{1}{2}r_1^2(\sin B + \sin C - \sin A) = \frac{r_1S}{2R} = 2r_1^2\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C,$$

with two similar expressions for Δ_2 and Δ_3.

Hence prove that $\Delta_1 + \Delta_2 + \Delta_3 - \Delta = 2S$.

16. O is the centre of the inscribed circle of a triangle ABC; O_1, O_2, O_3 are the centres of the escribed circles; r is the radius of the inscribed circle: shew that

$$r^3 . OO_1 . OO_2 . OO_3 = OA^2 . OB^2 . OC^2.$$

17. Let p, p', p'' be the perpendiculars dropped from the centre of the circle circumscribing a triangle, on the sides, and R the radius of the circle; prove that

$$R^3 - (p^2 + p'^2 + p''^2)R - 2pp'p'' = 0.$$ (See Art. 40.)

Examples on Chapter VII., Page 57.

1. Having given the sides a, b, and the angle A of a triangle, shew how to find the other parts; and discuss thoroughly the ambiguity that occurs.

Ex. $a=\sqrt{3}-1$, $b=\sqrt{3}+1$, $A=15^\circ$.

2. The two sides of a triangle are 25 and 29, and the cosine of the angle opposite the latter is $\frac{3}{5}$; required the third side.

3. Shew how to solve a triangle when two sides and the included angle are given.

Ex. Suppose the sides to be 10 and 20 and the included angle 60°.

4. Given $a=50$, $c=80$, $C=90^\circ$; solve the triangle.

5. Given $a=60$, $b=80$, $C=90^\circ$; solve the triangle.

6. Given $a=60$, $b=80$, $C=50^\circ\,25'\,30''$; solve the triangle.

7. Given $a=100$, $b=80$, $A=100^\circ$; solve the triangle.

8. Given $a=60$, $b=80$, $A=45^\circ$; solve the triangle.

9. Given $a=80$, $B=55^\circ\,24'$, $C=30^\circ\,36'$; solve the triangle.

10. Given $a=80$, $b=60$, $c=120$; solve the triangle and find its area.

11. Find R, r, and r_1 of the triangles in Examples 5, 7, and 10.

12. Solve Example 10 by dividing the triangle into two right-angled triangles. (See Art. 79.)

ANSWERS.

CHAPTER I.

1. $32^g\,45'$. 2. $33°\,25'\,46''\cdot236$. 3. $185''\cdot\dot{1}8\dot{5}$.

4. $\cdot50972$, $\cdot58345$. 5. $71°\cdot61972439$.

6. Circumference = 88 feet, Area = 616 square feet.

7. 234·375 square feet. 8. 5, 10.

9. The numbers of the sides of the pairs are

19, 342; 18, 162; 17, 102; 16, 72; 15, 54; 14, 42; 12, 27; 11, 22; 10, 18; 8, 12; 5, 6;

and the pairs with integer angles are

16, 72, with angles 175^g, $175°$;
10, 18, with angles 160^g, $160°$;
8, 12, with angles 150^g, $150°$;
5, 6, with angles 120^g, $120°$.

CHAPTER II.

1. $\cos\alpha = \pm\frac{3}{4}$, $\sin\beta = \pm\frac{5}{13}$, $\sec\gamma = \pm\sqrt{5}$, $\cos\gamma = \pm\frac{1}{\sqrt{5}}$, $\sin\gamma = \mp\frac{2}{\sqrt{5}}$, $\tan\delta = \pm 2\,.\,4$, $\cos\delta = \frac{5}{13}$, $\sin\delta = \pm\frac{12}{13}$.

2. $\cos\theta = \pm\sqrt{(1-a^2)}$, $\tan\theta = \pm\frac{a}{\sqrt{(1-a^2)}}$, &c.

3. $\sin\theta = \pm\sqrt{(1-a^2)}$, $\tan\theta = \frac{\pm\sqrt{(1-a^2)}}{a}$,

$$\sin\phi = \pm\frac{a}{\sqrt{(a^2+b^2)}},\quad \cos\phi = \pm\frac{b}{\sqrt{(a^2+b^2)}},\quad \cos\psi = \frac{1}{a},$$

$$\sin\psi = \frac{\pm\sqrt{(a^2-1)}}{a},\quad \tan\psi = \pm\sqrt{(a^2-1)}.$$

4. $\sin 330° = -\sin 30°$, $\cos 330° = \cos 30°$, $\sin 590° = -\sin 50°$, $\cos 590° = -\cos 50°$, $\sin(-370°) = -\sin 10°$, $\cos(-370°) = \cos 10°$, $\sin 1866° = \sin 66°$, $\cos 1866 = \cos 66°$.

5. (1) $\sin\theta = \frac{1}{2}$, (2) $\sin\theta = 0$ and $\cos\theta = \frac{1}{2}$,
(3) $\sin\theta = \frac{1}{2}$, (4) $\cos\theta = \pm\frac{1}{2}$.

7. Impossible unless $a = b$.

CHAPTER III.

2. (1) $\theta = n\pi + \frac{\pi}{4}$, (2) $\theta = n\pi + (-1)^n\frac{\pi}{2}$, (3) $\theta = 2n\pi$.

(4) $\theta = n\pi - (1)^n.\frac{\pi}{6}$, (5) $\theta = (2n \pm \frac{1}{2})\pi \pm a$.

3. $\theta = n\pi + (-1)^n.\frac{\pi}{6}$.

CHAPTERS IV. AND V.

1. $\tan A$. 2. $-\cot 3A \tan^2 A$. 3. $2\tan 2A$.

4. 0. 5. 1. 6. 1. 8. $\theta = \frac{\pi}{4} + 2n\pi \pm \frac{\pi}{6}$.

9. $2\theta = 2n\pi \pm \frac{\pi}{3}$.

10. $\sin 105° = \frac{1+\sqrt{3}}{2\sqrt{2}}$, $\cos 105° = \frac{1-\sqrt{3}}{2\sqrt{2}}$.

11. $\tan 165° = \sqrt{3} - 2$. 12. $\sin A = \frac{1}{2}$.

13. (1) $\theta = n\pi + (-1)^n\left(a - \frac{\pi}{2}\right)$ and

$$\theta = n\pi + (-1)^n\left(\frac{\pi}{2} - a \pm \frac{\pi}{3}\right).$$

(2) $\tan\theta = \cot a + \cot\beta$.

(3) $\tan(A - A') = \tan A + \tan B + \tan C$.

17. $\cos\frac{\pi}{5} = \frac{1}{4}(\sqrt{5}+1)$, $\cos\frac{2\pi}{5} = \frac{1}{4}(\sqrt{5}-1)$, &c.

18. $\theta = \frac{\pi}{12}$, common difference $= \sqrt{6}$.

CHAPTER VII.

1. $B = 105°$ or $75°$, $C = 60°$ or $90°$, and $c = \sqrt{6}$ or $2\sqrt{2}$.

2. 36. 3. $a = 10\sqrt{3}$, $B = 90°$, $C = 30°$.

4. $b = 62{\cdot}45$, $A = 38° \, 40' \, 56''$, $B = 51° \, 19' \, 4''$.

5. $c = 100$, $A = 36° \, 52' \, 11''{\cdot}6$, $B = 53° \, 7' \, 48''{\cdot}4$.

6. $A = 47° \, 54' \, 32''$, $B = 81° \, 39' \, 58''$, $c = 62{\cdot}3214$.

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12. Perpendicular on greatest side $= 35{\cdot}55126$ and segments of greatest side are $\frac{215}{3}$, $\frac{145}{3}$.

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